Symmetry Analysis of Heat and Wave Equations on Surfaces of Revolution

BY

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This thesis, written by **Kassimu Mpungu** under the direction of his thesis advisors and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

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To my beloved parents

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Table of Contents

List of Tables

List of Figures

Thesis Abstract

Name: Kassimu Mpungu

Title: Symmetry Analysis of Heat and Wave Equations on Surfaces of Revolution

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Lie symmetry method is a technique to find exact solutions of differential equations. One of the significant applications of Lie symmetry theory is to achieve a complete classification and analysis of Lie symmetries and symmetry reductions of differential equations.

This project is concerned with carrying out a complete symmetry analysis of the heat equation $\Delta u = u_t$ *and the wave equation* $\Delta u = u_{tt}$ *on a surface of revolution. Where* Δ *defines the Laplacian operator on the surface of revolution.*

For both the wave and the heat equations, the aim is to

- *Find the minimal symmetry algebra.*
- *Find all surfaces of revolution on which the equations have larger symmetry algebra and determine these algebras.*
- *Find some symmetry reductions and the corresponding exact solutions.*

ملخص الرسالة

الاسم: قاسم مبنغو

العنوان: التماثلات و الحلول الدقيقة لمعادلات الحرارة و الموجة على سطح الدوران

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طريقة لي للتماثلات هي تقنية لإيجاد الحلول الدقيقة للمعادلات التفاضلية. أحد أهم التطبيقات لنظرية لي في التماثلات هي الحصول على تصنيف و تحليل كامل لتماثلات لي و الإختزالات التماثلية للمعادلات التفاضلية.

تهدف هذه الرسالة إلى تحليل كامل لتماثلات معادلة الحرارة $u_t = \Delta u$ و معادلة الموجة $u_{tt} = \Delta u$ على سطح دوراني. حيث تمثل Δ معامل لابلاس على سطح دوراني.

لكل من معادلات الموجة و معادلات الحرارة، تهدف هذه الرسالة:

- إيجاد أصغر جبر تماثلي.
- إيجاد كل الأسطح الدورانية للمعادلات التي يكون عليها أكبر جبر تماثلي.
- إيجاد بعض الإختزالات التماثلية و الحلول الدقيقة المرتبطة بها.

Chapter 1

Introduction

Most physical processes naturally involve changes through series of states. This often makes it possible for the scientists to explicitly express such processes in terms of mathematical models. The mathematical modeling of most of these processes results into differential equations whose analytic solutions are in most cases hard to find. Therefore, investigations related to simplifications of differential equations and construction of their exact solutions become significant in the analysis of such physical processes. Lie symmetry method has proven to be a powerful technique for analyzing linear and non-linear ODEs and PDEs. It provides the most widely applicable technique to find exact solutions of differential equations and contains, as particular case cf. [38], many efficient methods for solving differential equations like separation of variables, traveling wave solutions, self-similar solutions and exponential self-similar solutions.

The classical Lie symmetry theory to study differential equations was developed by Sophus Lie more than a century ago. A modern treatment of the classical Lie symmetry theory was provided by Ovsiannikov [37]. Since the modern treatment by Ovsiannikov, the theory has substantially grown and has found wide spread uses. A large amount of literature about the classical Lie symmetry theory, its applications and its extensions is available, e.g. [2 6, 7, 8, 11, 17, 19, 20, 21, 22, 23, 32, 36, 37, and 41].

Some of the geometrically rich and physically significant classes of surfaces include the ruled surfaces, surfaces of revolution, tubular surfaces which are defined by space curves, as well as minimal surfaces which are important in the theory of soap films. These classes of surfaces possess a wide range of applications, for example, in computer graphics, digital design, architecture, engineering design, study of biological membranes, sheet metal based industries, the study of key objects in most nonlinear phenomena in physics and field theories etc.

Surfaces of revolution form a large class of surfaces, which are generated by rotating a plane curve about an axis. Hence, such surfaces naturally posses nice symmetry properties. This makes them and related problems an interesting area of contemporary research, and particularly of importance in the fields of physics, engineering, computer graphics and other disciplines involving models of physical processes with a natural symmetries. Well known examples of surfaces of revolution include cylinder, cone, sphere, hyperboloid, ellipsoid, Gabriel's horn, pseudosphere, torus, catenoid and tractoid.

A common phenomenon that appears in many fields like fluid mechanics, plasma physics, hydrodynamics and general relativity is the wave phenomena. Therefore, the studies related to exact solutions and properties of wave equations have remained of significant interest.

A series of recent papers [4, 12, 33, 34, 35] have been devoted to studying wave or heat equation, using symmetries, on specific cases of surfaces like sphere, torus, cone and hyperbolic space.

The aim of this work is to take a unified general approach by considering heat and wave equations on a general surface of revolution and investigate the group classification problem which, in general, consists of two main steps. The first step is finding the Lie symmetries of the heat and the wave equation on an arbitrary surface of revolution. The second step is determining all possible surfaces of revolution for which larger symmetry groups exist.

The first group classification problem was carried out by Ovsiannikov [37] who classified all forms of the non-linear heat equation. $u_t = (f(u)u_x)_x$. Since then, a number of articles on symmetry analysis and classification problem for non-linear PDEs have appeared in literature. For example Group properties of $u_{tt} = (f(u)u_x)_x$ were studied in [3], a study on Group classification, optimal system and optimal symmetry reductions of a class of Klein Gorden equations, Communications in nonlinear science and numerical simulation was carried in [5] whereas in [13] Symmetry classification and optimal systems of a non-linear wave equation was considered. A series of other papers include [10, 16, 18, 24, 25, 26, 27, 28, 29, 30, 39, 40, 42, and 43].

1.1. Some basic definitions from differential geometry

To formulate our problem clearly, we need to give some definitions from differential geometry [14], more specifically the definition of Laplacian on surfaces. In this section we define the Riemann metric or first fundamental form of surfaces, the Laplacian and Gaussian curvature of surfaces.

Definition:

Let $X(x, y)$ be a coordinate patch or parameterization of a surface M.

Define:

$$E = X_x \cdot X_x,\ \ F = X_x \cdot X_y,\ \ G = X_y \cdot X_y$$

Then,

$$g = ds^2 = Edx^2 + 2Fdxdy + Gdy^2$$

is called the first fundamental form or Riemannian metric of the surface M.

Classical notation of the Metric

Set

$$g_{11} = E = X_x \cdot X_x,\ g_{12} = g_{21} = F = X_x \cdot X_y,\ g_{22} = G = X_y \cdot X_y$$

Then it is often convenient to put the metric

$$g = ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$$

in the form of a symmetric matrix

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = (g_{ij})$$

Note that

$$\det(g) = g_{11}g_{22} - g_{12}^2$$

Therefore,

$$g^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{pmatrix} = \frac{1}{\det(g)}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} = (g^{ij})$$

Laplacian on a Surface

Consider a surface with a metric g. Then the Laplacian on the surface is defined as

$$\Delta u = \frac{1}{\sqrt{|\det(g)|}}\frac{\partial}{\partial x^i}\left(\sqrt{|\det(g)|}g^{ij}\frac{\partial u}{\partial x^j}\right)$$

where the summation is taken over repeated indices.

Gaussian curvature

Since the symmetry analysis of this thesis is carried out on surfaces of different curvatures, it is worth recalling the formula for calculating the Gaussian curvature of surfaces and specifically the surface of revolution [14].

Let $X(x,y)$ be a coordinate patch or parameterization of a surface M. Setting

$$E = X_x \cdot X_x,\ \ F = X_x \cdot X_y,\ \ G = X_y \cdot X_y,\ \ U = \frac{X_x \times X_y}{|X_x \times X_y|}$$

then the components of the second fundamental form are

$$l = X_{xx} \cdot U, \quad m = X_{xy} \cdot U, \quad n = X_{yy} \cdot U$$

and the Gaussian curvature of the surface M is given by

$$K = \frac{ln - m^2}{EG - F^2}$$

Consider a surface of revolution generated by a unit speed curve $\alpha(x) = (v(x), w(x))$ with a parameterized by the coordinate patch

$$X(x,y) = (v(x), w(x)\cos y, w(x)\sin y).$$

For such a surface of revolution

$$X_x = (v', w'\cos y, w'\sin y),\ \ X_y = (0, -w\sin y, w\cos y)$$

and

$$X_x \times X_y = (ww', -v'w\cos x, -v'w\sin x)$$

Hence

$$U = (w', -v'\cos x, -v'\sin x)$$

Also, the second partial derivatives are

$$X_{xx} = (v'', w''\cos y, w''\sin y),\ \ X_{xy} = (0, -w'\sin y, w'\cos y)$$

and

$$X_{yy} = (0, -w\cos y, -w\sin y)$$

Thus, since $\alpha(x) = (v(x), w(x))$ is a unit speed curve, we have

$$E = v'^2 + w'^2 = 1, \qquad F = 0, \qquad G = w^2$$

and

$$l = v''w' - w''v', \qquad m = 0, \qquad n = v'w$$

Finally, the Gaussian curvature is computed to be

$$K = \frac{(v''w' - w''v')v'}{w(x)}$$

Theorem 1.1

The Gaussian curvature K of surface of revolution generated by a unit speed curve $\alpha(x) = (v(x), w(x))$ satisfies the equation.

$$w''(x) + Kw(x) = 0$$

Proof;

For a unit speed curve $\alpha(x) = (v(x), w(x))$,

$$v'^2 + w'^2 = 1$$

$$2v'v'' + 2w'w'' = 0$$

$$v'v'' = -w'w''$$

$$Kw(x) = (w'v''v' - w''v'^2)$$

$$= -(w'^2 w'' + w''v'^2)$$

$$= -w''(x)$$

$$w''(x) + Kw(x) = 0.$$

1.2. Problem formulation and main results.

On the surface of revolution parameterized by the coordinate patch $X(x,y) = (v(x), w(x)\cos y, w(x)\sin y)$ generated by revolving a unit speed curve $\alpha(x) = (v(x), w(x))$ as shown above, the metric is given by

$$g = ds^2 = dx^2 + w(x)^2 dy^2$$

so that the expression for the Laplacian becomes

$$\Delta u = \frac{w'(x)}{w(x)}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{w(x)^2}\frac{\partial^2 u}{\partial y^2}$$

However, for regularity of the coordinate patch $X(x,y)$ we must have $w(x) > 0$. i.e. $w(x)$ remains in upper half of its plane. To ensure this, we let $w(x) = e^{f(x)}$ for some smooth function f. We shall refer to f as a determining function since coordinate patch $X(x,y)$ entirely depends on f. Consequently the metric becomes

$$g = ds^2 = dx^2 + e^{2f(x)}dy^2$$

It then follows immediately that the Laplacian takes the form

$$\Delta u = f'(x)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + e^{-2f(x)}\frac{\partial^2 u}{\partial y^2}$$

1.2.1. Heat equation on a surface of revolution.

The heat equation on a surface of revolution parameterized by the coordinate patch $X(x,y) = (v(x),\ e^{f(x)}\cos y,\ e^{f(x)}\sin y)$ is given by $u_t = \Delta u$

where Δ denotes the Laplacian operator defined in the previous section. Hence the heat equation on the surface of revolution becomes

$$u_t = f'(x)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + e^{-2f(x)}\frac{\partial^2 u}{\partial y^2} \tag{1.1}$$

The symmetry analysis of equation (1.1) is carried out in Chapter 4, where the following result is proved.

Theorem 4.1

The minimal symmetry algebra of heat equation

$$u_t = f'(x)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + e^{-2f(x)}\frac{\partial^2 u}{\partial y^2}$$

on a surface of revolution is generated by

$$X_1 = \frac{\partial}{\partial y},\quad X_2 = \frac{\partial}{\partial t},\quad X_3 = u\frac{\partial}{\partial u}$$

and is obtained for an arbitrary determining function f. The larger symmetry algebra exists in the cases given in the table 1.1 below.

Table 1.1:

Determining functions for which larger symmetry algebras exist for the heat equation on a surface of revolution

$f(x)$	Description of the surface	Number of extra symmetries
c	Cylinder of radius e^c	6
$ax+b$	Pseudosphere or tractoid	2
$\ln\|ax+b\|$	$a=1$; Plane, $a\neq 1$; Cone	6
$\ln\|a\cos(bx)\|$	• $ab=1$; sphere of radius a • $ab<1$; Surface of a spindle type • $ab>1$; Surface of a bulge type	2
$\ln\|a\sinh(bx)\|$	Surface of a conic type	2
$\ln\|a\cosh(bx)\|$	Hyperboloid of one sheet	2
$\ln[b(x-c)^a]$; $b>0,\ x>c,\ a\neq 0,1$	Surfaces of revolution generated by a unit speed curve $\alpha(x)=(v(x),b(x-c)^a)$ For example • $a=\frac{1}{2},\ b=1,\ c=0$; Paraboloid • $a=-1,\ b=1\ c=0$; Gabriel's Horn	1

1.2.2. Wave equation on a surface of revolution.

The wave equation on a surface of revolution parameterized by the coordinate patch $X(x,y)=(v(x),\ e^{f(x)}\cos y,\ e^{f(x)}\sin y)$ is given by $u_{tt}=\Delta u$

where Δ denotes the Laplacian operator defined in section 1.1.

Hence the wave equation on the surface of revolution becomes

$$u_{tt}=f'(x)\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}+e^{-2f(x)}\frac{\partial^2 u}{\partial y^2} \tag{1.2}$$

The symmetry analysis of equation (1.2) is carried out in Chapter 5, where the following result is proved.

Theorem 5.1

The minimal symmetry algebra of wave equation

$$u_{tt}=f'(x)\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}+e^{-2f(x)}\frac{\partial^2 u}{\partial y^2}$$

on a surface of revolution is generated by

$$X_1=\frac{\partial}{\partial y},\quad X_2=\frac{\partial}{\partial t},\quad X_3=u\frac{\partial}{\partial u}$$

and is obtained for an arbitrary determining function f. *The larger symmetry algebra exists in the cases given in the table 1.2 below.*

Table 1.2:

Determining functions for which larger symmetry algebras exist for the wave equation on a surface of revolution.

$f(x)$	Description of the surface	Number of extra symmetries
c	Cylinder of radius e^c	8
$ax+b$	Pseudosphere or tractoid	2
$\ln\lvert ax+b \rvert$	$a=1$; Plane, $a \neq 1$; Cone	8
$\ln\lvert a\cos(bx) \rvert$	• $ab=1$; sphere of radius a • $ab<1$; Surface of a spindle type • $ab>1$; Surface of a bulge type	2
$\ln\lvert a\sinh(bx) \rvert$	Surface of a conic type	2
$\ln\lvert a\cosh(bx) \rvert$	Hyperboloid of one sheet	2
$\ln[b(x-c)^a]$; $b>0,\ x>c,\ a\neq 0,1$	Surfaces of revolution generated by a unit speed curve $\boldsymbol{\alpha}(x)=(v(x), b(x-c)^a)$ For example • $a=\frac{1}{2},\ b=1,\ c=0$; Paraboloid • $a=-1,\ b=1\ \ c=0$; Gabriel's Horn	1

Chapter 2

Lie symmetry method for partial differential equations

This chapter focuses on basic ideas of Lie symmetry method which serves as the basis for the research results presented in chapters 3, 4 and 5. The main objective is to give a short review of the standard background in Lie symmetry method for PDEs. In particular this chapter is devoted to discussion of symmetries of PDEs, the prolongations of their generators and the method of finding symmetries of PDEs. Most of the proofs in this review are not given because the fundamental results of Lie symmetry methods are well established and have turned to be standard in the recent literature. However, the most essential details are presented.

Throughout the whole chapter, we shall restrict our work to second order PDEs with one dependent variable u and three independent variables x, y, t for the sake of simplicity. This will not in any case affect our results in the chapters 3, 4 and 5 of our work since the PDEs involved all belong to this class of PDEs. A comprehensive account of the subject of Lie symmetry for general PDEs is contained in many standard books on the topic cf. [2, 6, 7, 8, 11, 17, 19, 20, 21, 22, 32, 36, 37].

Consider a PDE

$$F(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{xy},u_{xt},u_{yy},u_{yt},u_{tt})=0 \tag{2.1}$$

A one parameter group of transformations.

$$x^* = g(x,y,t,u,\varepsilon),\ y^* = h(x,y,t,u,\varepsilon),\ t^* = p(x,y,t,u,\varepsilon),\ u^* = q(x,y,t,u,\varepsilon) \tag{2.2}$$

is a symmetry of PDE (2.1) if the PDE (2.1) is invariant under the transformation (2.2.), i.e. after change of variables

$$(x,y,t,u)\to(x^*,y^*,t^*,u^*)$$

we get

$$F(x^*,y^*,t^*,u^*,u^*_x,u^*_y,u^*_t,u^*_{xx},u^*_{xy},u^*_{xt},u^*_{yy},u^*_{yt},u^*_{tt})=0$$

We shall represent the functions g, h, p and q via their Taylor series expansion with respect to the parameter ε in the neighborhood of $\varepsilon = 0$ and write the infinitesimal form of transformation (2.2) as follows

$$x^* = x+\varepsilon\xi(x,y,t,u)+O(\varepsilon^2)$$
$$y^* = y+\varepsilon\vartheta(x,y,t,u)+O(\varepsilon^2)$$
$$t^* = t+\varepsilon\tau(x,y,t,u)+O(\varepsilon^2)$$
$$u^* = u+\varepsilon\varphi(x,y,t,u)+O(\varepsilon^2)$$

where

$$\xi(x,y,t,u)=g_\varepsilon\Big|_{\varepsilon=0},\ \vartheta(x,y,t,u)=h_\varepsilon\Big|_{\varepsilon=0},\ \tau(x,y,t,u)=p_\varepsilon\Big|_{\varepsilon=0},\ \phi(x,y,t,u)=q_\varepsilon\Big|_{\varepsilon=0}.$$

The tangent vector field $(\xi,\vartheta,\tau,\phi)$ can be written in terms of the first-order differential operator.

$$X=\xi(x,y,t,u)\frac{\partial}{\partial x}+\vartheta(x,y,t,u)\frac{\partial}{\partial y}+\tau(x,y,t,u)\frac{\partial}{\partial t}+\varphi(x,y,t,u)\frac{\partial}{\partial u} \tag{2.3}$$

The differential operator (2.3) is known as *symmetry* or *infinitesimal* operator or generator. These terms will be used interchangeably.

2.1. Prolongations of infinitesimal generators of symmetries of PDEs

The symmetry operator (2.3) provides information on how the variables x,y,t and u are transformed. However this information is not enough. We also need information on how the partial derivatives of u are transformed.

In this section we discuss how to prolong infinitesimal generators and the essence of this to exhaustively obtain all the information on how the variables of the PDE (2.1) and derivatives of u are transformed.

The derivation of prolongation formulas is restricted to symmetries of 2^{nd} order PDEs as the PDE (2.1) is of second order.

Now we write the transformation formulas for the partial derivatives of u corresponding to the point transformation. It is convenient to use the operator of total differentiation.

Definition:

Consider the function

$$F(x,y,t,u(x,y,t),g_1(x,y,t),g_2(x,y,t),\ldots,g_n(x,y,t))=0$$

The total differentiation operators with respect to x, y and t are defined respectively as

$$D_x=\frac{\partial}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial g_1}{\partial x}\frac{\partial}{\partial g_1}+\cdots+\frac{\partial g_n}{\partial x}\frac{\partial}{\partial g_n}$$

$$D_y=\frac{\partial}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial}{\partial u}+\frac{\partial g_1}{\partial y}\frac{\partial}{\partial g_1}+\cdots+\frac{\partial g_n}{\partial y}\frac{\partial}{\partial g_n}$$

$$D_t=\frac{\partial}{\partial t}+\frac{\partial u}{\partial t}\frac{\partial}{\partial u}+\frac{\partial g_1}{\partial t}\frac{\partial}{\partial g_1}+\cdots+\frac{\partial g_n}{\partial t}\frac{\partial}{\partial g_n}$$

As an example, we consider the PDE (2.1), the total differentiation operator with respect to x, y and t take the form.

$$D_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xy}\frac{\partial}{\partial u_y}+u_{xt}\frac{\partial}{\partial u_t}+\cdots+u_{xtt}\frac{\partial}{\partial u_{tt}}$$

$$D_y=\frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}+u_{xy}\frac{\partial}{\partial u_x}+u_{yy}\frac{\partial}{\partial u_y}+u_{ty}\frac{\partial}{\partial u_t}+\cdots+u_{ytt}\frac{\partial}{\partial u_{tt}}$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{yt} \frac{\partial}{\partial u_y} + u_{tt} \frac{\partial}{\partial u_t} + \cdots + u_{ttt} \frac{\partial}{\partial u_{tt}}$$

First Prolongation of X

Consider the symmetry operator (2.3)

$$X = \xi(x,y,t,u)\frac{\partial}{\partial x} + \vartheta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \varphi(x,y,t,u)\frac{\partial}{\partial u}$$

as already clarified this is equivalent to the infinitesimal transformation

$$x^* = x + \varepsilon\xi(x,y,t,u) + O(\varepsilon^2) \tag{2.4}$$

$$y^* = y + \varepsilon\vartheta(x,y,t,u) + O(\varepsilon^2) \tag{2.5}$$

$$t^* = t + \varepsilon\tau(x,y,t,u) + O(\varepsilon^2) \tag{2.6}$$

$$u^* = u + \varepsilon\varphi(x,y,t,u) + O(\varepsilon^2) \tag{2.7}$$

We want to find the transformation of the first order partial derivatives of u with respect to x, y and t. i.e. we need to obtain the functions

$$\eta^{[x]}(x,y,t,u,u_x,u_y,u_t),\ \eta^{[y]}(x,y,t,u,u_x,u_y,u_t) \text{ and } \eta^{[t]}(x,y,t,u,u_x,u_y,u_t)$$

such that

$$u^*_{x^*} = u_x + \varepsilon\eta^{[x]}(x,y,t,u,u_x,u_y,u_t) + O(\varepsilon^2) \tag{2.8}$$

$$u^*_{y^*} = u_y + \varepsilon\eta^{[y]}(x,y,t,u,u_x,u_y,u_t) + O(\varepsilon^2) \tag{2.9}$$

$$u^*_{t^*} = u_t + \varepsilon\eta^{[t]}(x,y,t,u,u_x,u_y,u_t) + O(\varepsilon^2) \tag{2.10}$$

From Eq(2.4), we have

$$\begin{aligned} dx^* &= dx + \varepsilon d\xi + O(\varepsilon^2) \\ &= dx + \varepsilon[\xi_x dx + \xi_y dy + \xi_t dt + \xi_u du] + O(\varepsilon^2) \\ &= dx + \varepsilon[\xi_x dx + \xi_y dy + \xi_t dt + \xi_u(\tfrac{\partial u}{\partial x}dx + \tfrac{\partial u}{\partial y}dy + \tfrac{\partial u}{\partial t}dt)] + O(\varepsilon^2) \\ &= [1 + \varepsilon(\xi_x + \xi_u \tfrac{\partial u}{\partial x})]dx + \varepsilon[\xi_y + \xi_u \tfrac{\partial u}{\partial y}]dy + \varepsilon[\xi_t + \xi_u \tfrac{\partial u}{\partial t}]dt + O(\varepsilon^2) \end{aligned}$$

This implies that

$$dx^* = [1 + \varepsilon D_x\xi]dx + \varepsilon[D_y\xi]dy + \varepsilon[D_t\xi]dt + O(\varepsilon^2) \tag{2.11}$$

Similarly from equations (2.5) and (2.6), we respectively have

$$dy^* = \varepsilon[D_x\vartheta]dx + [1 + \varepsilon D_y\vartheta]dy + \varepsilon[D_t\vartheta]dt + O(\varepsilon^2) \tag{2.12}$$

$$dt^* = \varepsilon[D_x\tau]dx + \varepsilon[D_y\tau]dy + [1 + \varepsilon D_t\tau]dt + O(\varepsilon^2) \tag{2.13}$$

From equation (2.7), we note that

$$\begin{aligned} du^* &= du + \varepsilon d\varphi + O(\varepsilon^2) \\ &= \tfrac{\partial u}{\partial x}dx + \tfrac{\partial u}{\partial y}dy + \tfrac{\partial u}{\partial t}dt + \varepsilon[\varphi_x dx + \varphi_y dy + \varphi_t dt + \varphi_u(\tfrac{\partial u}{\partial x}dx + \tfrac{\partial u}{\partial y}dy + \tfrac{\partial u}{\partial t}dt)] + O(\varepsilon^2) \\ &= [\tfrac{\partial u}{\partial x} + \varepsilon(\varphi_x + \varphi_u \tfrac{\partial u}{\partial x})]dx + [\tfrac{\partial u}{\partial y} + \varepsilon(\varphi_y + \varphi_u \tfrac{\partial u}{\partial y})]dy + [\tfrac{\partial u}{\partial t} + \varepsilon(\varphi_t + \varphi_u \tfrac{\partial u}{\partial t})]dt + O(\varepsilon^2) \end{aligned}$$

This implies that

$$du^* = [\tfrac{\partial u}{\partial x} + \varepsilon D_x\varphi]dx + [\tfrac{\partial u}{\partial y} + \varepsilon D_y\varphi]dy + [\tfrac{\partial u}{\partial t} + \varepsilon D_t\varphi]dt + O(\varepsilon^2) \tag{2.14}$$

Also

$$u^* = u^*(x^*, y^*, t^*)$$

This implies that

$$du^* = \tfrac{\partial u^*}{\partial x^*}dx^* + \tfrac{\partial u^*}{\partial y^*}dy^* + \tfrac{\partial u^*}{\partial t^*}dt^* \tag{2.15}$$

Using equations (2.11)-(2.14) in Eq. (2.15), and organizing give

$$\begin{aligned} &[\tfrac{\partial u}{\partial x} + \varepsilon D_x\varphi]dx + [\tfrac{\partial u}{\partial y} + \varepsilon D_y\varphi]dy + [\tfrac{\partial u}{\partial t} + \varepsilon D_t\varphi]dt + O(\varepsilon^2) \\ &= \left\{\tfrac{\partial u^*}{\partial x^*}[1 + \varepsilon D_x\xi] + \varepsilon\tfrac{\partial u^*}{\partial y^*}[D_x\vartheta] + \varepsilon\tfrac{\partial u^*}{\partial t^*}[D_x\tau]\right\}dx \\ &+ \left\{\varepsilon\tfrac{\partial u^*}{\partial x^*}[D_y\xi] + \varepsilon\tfrac{\partial u^*}{\partial t^*}[D_y\tau] + \tfrac{\partial u^*}{\partial y^*}[1 + \varepsilon D_y\vartheta]\right\}dy \\ &+ \left\{\varepsilon\tfrac{\partial u^*}{\partial x^*}[D_t\xi] + \varepsilon\tfrac{\partial u^*}{\partial y^*}[D_t\vartheta] + \tfrac{\partial u^*}{\partial t^*}[1 + \varepsilon D_t\tau]\right\}dt + O(\varepsilon^2) \end{aligned}$$

But since dx dy and dt are linearly independent, the above relation implies that

$$\begin{aligned} \tfrac{\partial u}{\partial x} + \varepsilon D_x\varphi &= \tfrac{\partial u^*}{\partial x^*}[1 + \varepsilon D_x\xi] + \varepsilon\tfrac{\partial u^*}{\partial y^*}[D_x\vartheta] + \varepsilon\tfrac{\partial u^*}{\partial t^*}[D_x\tau] \\ \tfrac{\partial u}{\partial y} + \varepsilon D_y\varphi &= \varepsilon\tfrac{\partial u^*}{\partial x^*}[D_y\xi] + \varepsilon\tfrac{\partial u^*}{\partial t^*}[D_y\tau] + \tfrac{\partial u^*}{\partial y^*}[1 + \varepsilon D_y\vartheta] \\ \tfrac{\partial u}{\partial t} + \varepsilon D_t\varphi &= \varepsilon\tfrac{\partial u^*}{\partial x^*}[D_t\xi] + \varepsilon\tfrac{\partial u^*}{\partial y^*}[D_t\vartheta] + \tfrac{\partial u^*}{\partial t^*}[1 + \varepsilon D_t\tau] \end{aligned} \tag{2.16}$$

Next we express the system (2.16) as a matrix below

$$\begin{pmatrix} \tfrac{\partial u}{\partial x} + \varepsilon D_x\varphi \\ \tfrac{\partial u}{\partial y} + \varepsilon D_y\varphi \\ \tfrac{\partial u}{\partial t} + \varepsilon D_t\varphi \end{pmatrix} = \begin{pmatrix} 1 + \varepsilon D_x\xi & \varepsilon D_x\vartheta & \varepsilon D_x\tau \\ \varepsilon D_y\xi & 1 + \varepsilon D_y\vartheta & \varepsilon D_y\tau \\ \varepsilon D_t\xi & \varepsilon D_t\vartheta & 1 + \varepsilon D_t\tau \end{pmatrix} \begin{pmatrix} \tfrac{\partial u^*}{\partial x^*} \\ \tfrac{\partial u^*}{\partial y^*} \\ \tfrac{\partial u^*}{\partial t^*} \end{pmatrix} \tag{2.17}$$

If we set

$$B = \begin{pmatrix} D_x\xi & D_x\vartheta & D_x\tau \\ D_y\xi & D_y\vartheta & D_y\tau \\ D_t\xi & D_t\vartheta & D_t\tau \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1+\varepsilon D_x\xi & \varepsilon D_x\vartheta & \varepsilon D_x\tau \\ \varepsilon D_y\xi & 1+\varepsilon D_y\vartheta & \varepsilon D_y\tau \\ \varepsilon D_t\xi & \varepsilon D_t\vartheta & 1+\varepsilon D_t\tau \end{pmatrix},$$

then we have

$$A = I + \varepsilon B$$

This implies that

$$A^{-1} = (I+\varepsilon B)^{-1} = I - \varepsilon B + O(\varepsilon^2)$$

Then, from equation (2.17), we get

$$\begin{pmatrix} \frac{\partial u^*}{\partial x^*} \\ \frac{\partial u^*}{\partial y^*} \\ \frac{\partial u^*}{\partial t^*} \end{pmatrix} = A^{-1} \begin{pmatrix} \frac{\partial u}{\partial x} + \varepsilon D_x\varphi \\ \frac{\partial u}{\partial y} + \varepsilon D_y\varphi \\ \frac{\partial u}{\partial t} + \varepsilon D_t\varphi \end{pmatrix} + O(\varepsilon^2)$$

this is equivalent to

$$\begin{pmatrix} \frac{\partial u^*}{\partial x^*} \\ \frac{\partial u^*}{\partial y^*} \\ \frac{\partial u^*}{\partial t^*} \end{pmatrix} = (I-\varepsilon B) \begin{pmatrix} \frac{\partial u}{\partial x} + \varepsilon D_x\varphi \\ \frac{\partial u}{\partial y} + \varepsilon D_y\varphi \\ \frac{\partial u}{\partial t} + \varepsilon D_t\varphi \end{pmatrix} + O(\varepsilon^2)$$

Simplifying using (2.8)-(2.10) the above gives

$$\begin{pmatrix} \frac{\partial u}{\partial x} + \varepsilon\eta^{[x]}(x,y,t,u,u_x,u_y,u_t) + O(\varepsilon^2) \\ \frac{\partial u}{\partial y} + \varepsilon\eta^{[y]}(x,y,t,u,u_x,u_y,u_t) + O(\varepsilon^2) \\ \frac{\partial u}{\partial t} + \varepsilon\eta^{[t]}(x,y,t,u,u_x,u_y,u_t) + O(\varepsilon^2) \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial t} \end{pmatrix} + \varepsilon \begin{pmatrix} D_x\varphi \\ D_y\varphi \\ D_t\varphi \end{pmatrix} - \varepsilon B \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial t} \end{pmatrix} + O(\varepsilon^2)$$

this implies that

$$\begin{pmatrix} \eta^{[x]} \\ \eta^{[y]} \\ \eta^{[t]} \end{pmatrix} = \begin{pmatrix} D_x\varphi \\ D_y\varphi \\ D_t\varphi \end{pmatrix} - \begin{pmatrix} D_x\xi & D_x\vartheta & D_x\tau \\ D_y\xi & D_y\vartheta & D_y\tau \\ D_t\xi & D_t\vartheta & D_t\tau \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial t} \end{pmatrix} \tag{2.19}$$

Next, from the Eq(2.19) we write $\eta^{[x]}, \eta^{[y]}$ and $\eta^{[t]}$ in term of ξ, ϑ, τ and φ gives

$$\begin{aligned} \eta^{[x]} &= D_x\varphi - (u_x D_x\xi + u_y D_x\vartheta + u_t D_x\tau) \\ &= \varphi_x + (\varphi_u - \xi_x)u_x - \vartheta_x u_y - \tau_x u_t - \xi_u u_x^2 - \vartheta_u u_x u_y - \tau_u u_x u_t \end{aligned} \tag{2.20}$$

Similarly

$$\eta^{[y]} = \varphi_y - \xi_y u_x + (\varphi_u - \vartheta_y)u_y - \tau_y u_t - \xi_u u_x u_y - \vartheta_u u_y^2 - \tau_u u_y u_t \tag{2.21}$$

$$\eta^{[t]} = \varphi_t - \xi_t u_x - \vartheta_t u_y + (\varphi_u - \tau_t)u_t - \xi_u u_x u_t - \vartheta_u u_y u_t - \tau_u u_t^2 \qquad (2.22)$$

We then write down the first prolongation as follows

$$X^{[1]} = X + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[y]}\frac{\partial}{\partial u_y} + \eta^{[t]}\frac{\partial}{\partial u_t} \qquad (2.23)$$

where $\eta^{[x]}, \eta^{[y]}$ and $\eta^{[t]}$ are respectively given by (2.20), (2.21) and (2.22).

Second Prolongation of X

Consider the first prolongation of the operator X given by Eq(2.23)

$$X^{[1]} = X + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[y]}\frac{\partial}{\partial u_y} + \eta^{[t]}\frac{\partial}{\partial u_t}$$

We need to find the transformation of the second order partial derivatives of u with respect to x, y and t. i.e. we need to obtain the functions

$$\eta^{[xx]},\ \eta^{[xy]},\ \eta^{[xt]},\ \eta^{[yy]},\ \eta^{[yt]} \text{ and } \eta^{[tt]}$$

such that

$$u^*_{x^*x^*} = u_{xx} + \varepsilon\eta^{[xx]}(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{xy},u_{xt},u_{yy},u_{yt},u_{tt}) + O(\varepsilon^2) \qquad (2.24)$$

$$u^*_{x^*y^*} = u_{xy} + \varepsilon\eta^{[xy]}(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{xy},u_{xt},u_{yy},u_{yt},u_{tt}) + O(\varepsilon^2) \qquad (2.25)$$

$$u^*_{x^*t^*} = u_{xt} + \varepsilon\eta^{[xt]}(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{xy},u_{xt},u_{yy},u_{yt},u_{tt}) + O(\varepsilon^2) \qquad (2.26)$$

$$u^*_{y^*y^*} = u_{yy} + \varepsilon\eta^{[yy]}(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{xy},u_{xt},u_{yy},u_{yt},u_{tt}) + O(\varepsilon^2) \qquad (2.27)$$

$$u^*_{y^*t^*} = u_{yt} + \varepsilon\eta^{[yt]}(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{xy},u_{xt},u_{yy},u_{yt},u_{tt}) + O(\varepsilon^2) \qquad (2.28)$$

$$u^*_{t^*t^*} = u_{tt} + \varepsilon\eta^{[tt]}(x,y,t,u,u_x,u_y,u_t,u_{xx},u_{xy},u_{xt},u_{yy},u_{yt},u_{tt}) + O(\varepsilon^2) \qquad (2.29)$$

In our discussion however, we only restrict ourselves to obtaining the terms $\eta^{[xx]}$, $\eta^{[yy]}$ and $\eta^{[tt]}$ since the other terms are not required in solving our problem.

From Eq(2.8) we note

$$du^*_{x^*} = du_x + \varepsilon d\eta^{[x]} + O(\varepsilon^2)$$

$$
\begin{aligned}
&= \tfrac{\partial u_x}{\partial x}dx + \tfrac{\partial u_x}{\partial t}dt + \tfrac{\partial u_x}{\partial y}dy + \varepsilon[\tfrac{\partial \eta^{[x]}}{\partial x}dx + \tfrac{\partial \eta^{[x]}}{\partial t}dt + \tfrac{\partial \eta^{[x]}}{\partial y}dy + \tfrac{\partial \eta^{[x]}}{\partial u}du + \tfrac{\partial \eta^{[x]}}{\partial u_x}du_x \\
&\quad + \tfrac{\partial \eta^{[x]}}{\partial u_y}du_y + \tfrac{\partial \eta^{[x]}}{\partial u_t}du_t] + O(\varepsilon^2) \\
&= [\tfrac{\partial u_x}{\partial x} + \varepsilon(\tfrac{\partial \eta^{[x]}}{\partial x} + \tfrac{\partial \eta^{[x]}}{\partial u}\tfrac{\partial u}{\partial x} + \tfrac{\partial \eta^{[x]}}{\partial u_x}\tfrac{\partial u_x}{\partial x} + \tfrac{\partial \eta^{[x]}}{\partial u_y}\tfrac{\partial u_y}{\partial x} + \tfrac{\partial \eta^{[x]}}{\partial u_t}\tfrac{\partial u_t}{\partial x})]dx \\
&\quad + [\tfrac{\partial u_x}{\partial y} + \varepsilon(\tfrac{\partial \eta^{[x]}}{\partial y} + \tfrac{\partial \eta^{[x]}}{\partial u}\tfrac{\partial u}{\partial y} + \tfrac{\partial \eta^{[x]}}{\partial u_x}\tfrac{\partial u_x}{\partial y} + \tfrac{\partial \eta^{[x]}}{\partial u_y}\tfrac{\partial u_y}{\partial y} + \tfrac{\partial \eta^{[x]}}{\partial u_t}\tfrac{\partial u_t}{\partial y})]dy \\
&\quad + [\tfrac{\partial u_x}{\partial t} + \varepsilon(\tfrac{\partial \eta^{[x]}}{\partial t} + \tfrac{\partial \eta^{[x]}}{\partial u}\tfrac{\partial u}{\partial t} + \tfrac{\partial \eta^{[x]}}{\partial u_x}\tfrac{\partial u_x}{\partial t} + \tfrac{\partial \eta^{[x]}}{\partial u_y}\tfrac{\partial u_y}{\partial t} + \tfrac{\partial \eta^{[x]}}{\partial u_t}\tfrac{\partial u_t}{\partial t})]dt + O(\varepsilon^2)
\end{aligned}
$$

which implies that

$$
du^*_{x^*} = (\tfrac{\partial u_x}{\partial x} + \varepsilon D_x\eta^{[x]})dx + (\tfrac{\partial u_x}{\partial y} + \varepsilon D_y\eta^{[x]})dy + (\tfrac{\partial u_x}{\partial t} + \varepsilon D_t\eta^{[x]})dt + O(\varepsilon^2) \tag{2.30}
$$

Using equations (2.11), (2.12) and (2.13) in the following formula

$$
du^*_{x^*} = \tfrac{\partial u^*_{x^*}}{\partial x^*}dx^* + \tfrac{\partial u^*_{x^*}}{\partial y^*}dy^* + \tfrac{\partial u^*_{x^*}}{\partial t^*}dt^*
$$

gives

$$
\begin{aligned}
du^*_{x^*} &= u^*_{x^*x^*}([1+\varepsilon D_x\xi]dx + \varepsilon[D_y\xi]dy + \varepsilon[D_t\xi]dt) \\
&\quad + u^*_{x^*y^*}(\varepsilon[D_x\vartheta]dx + [1+\varepsilon D_y\vartheta]dy + \varepsilon[D_t\vartheta]dt) \\
&\quad + u^*_{x^*t^*}(\varepsilon[D_x\tau]dx + \varepsilon[D_y\tau]dy + [1+\varepsilon D_t\tau]dt) + O(\varepsilon^2)
\end{aligned}
$$

Using Eq. (2.30) and the independence of dx, dy and dt imply that

$$
\begin{aligned}
\tfrac{\partial u_x}{\partial x} + \varepsilon D_x\eta^{[x]} &= u^*_{x^*x^*}[1+\varepsilon D_x\xi] + \varepsilon u^*_{x^*y^*}[D_x\vartheta] + \varepsilon u^*_{x^*t^*}[D_x\tau] \\
\tfrac{\partial u_x}{\partial y} + \varepsilon D_y\eta^{[x]} &= \varepsilon u^*_{x^*x^*}[D_y\xi] + u^*_{x^*y^*}[1+\varepsilon D_y\vartheta] + \varepsilon u^*_{x^*t^*}[D_y\tau] \\
\tfrac{\partial u_x}{\partial t} + \varepsilon D_t\eta^{[x]} &= \varepsilon u^*_{x^*x^*}[D_t\xi] + \varepsilon u^*_{x^*y^*}[D_t\vartheta] + u^*_{x^*t^*}[1+\varepsilon D_t\tau]
\end{aligned}
$$

we now express the above in a matrix form

$$
\begin{aligned}
\begin{pmatrix} u_{xx} + \varepsilon D_x\eta^{[x]} \\ u_{xy} + \varepsilon D_y\eta^{[x]} \\ u_{xt} + \varepsilon D_t\eta^{[x]} \end{pmatrix} &= \begin{pmatrix} 1+\varepsilon D_x\xi & \varepsilon D_x\vartheta & \varepsilon D_x\tau \\ \varepsilon D_y\xi & 1+\varepsilon D_y\vartheta & \varepsilon D_y\tau \\ \varepsilon D_t\xi & \varepsilon D_t\vartheta & 1+\varepsilon D_t\tau \end{pmatrix} \begin{pmatrix} u^*_{x^*x^*} \\ u^*_{x^*y^*} \\ u^*_{x^*t^*} \end{pmatrix} + O(\varepsilon^2) \\
&= A \begin{pmatrix} u^*_{x^*x^*} \\ u^*_{x^*y^*} \\ u^*_{x^*t^*} \end{pmatrix} + O(\varepsilon^2)
\end{aligned}
$$

$$= (I + \varepsilon B)\begin{pmatrix} u^*_{x^*x^*} \\ u^*_{x^*y^*} \\ u^*_{x^*t^*} \end{pmatrix} + O(\varepsilon^2)$$

this implies that

$$\begin{pmatrix} u^*_{x^*x^*} \\ u^*_{x^*y^*} \\ u^*_{x^*t^*} \end{pmatrix} = (I + \varepsilon B)^{-1}\begin{pmatrix} u_{xx} + \varepsilon D_x \eta^{[x]} \\ u_{xy} + \varepsilon D_y \eta^{[x]} \\ u_{xt} + \varepsilon D_t \eta^{[x]} \end{pmatrix} + O(\varepsilon^2)$$

$$= (I - \varepsilon B)\begin{pmatrix} u_{xx} + \varepsilon D_x \eta^{[x]} \\ u_{xy} + \varepsilon D_y \eta^{[x]} \\ u_{xt} + \varepsilon D_t \eta^{[x]} \end{pmatrix} + O(\varepsilon^2)$$

$$= \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{xt} \end{pmatrix} + \varepsilon\left[\begin{pmatrix} D_x \eta^{[x]} \\ D_y \eta^{[x]} \\ D_t \eta^{[x]} \end{pmatrix} - B\begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{xt} \end{pmatrix}\right] + O(\varepsilon^2) \qquad (2.31)$$

Now comparing the equation (2.24), (2.25), (2.26) and (2.31), we note that

$$\begin{pmatrix} \eta^{[xx]} \\ \eta^{[xy]} \\ \eta^{[xt]} \end{pmatrix} = \begin{pmatrix} D_x \eta^{[x]} \\ D_y \eta^{[x]} \\ D_t \eta^{[x]} \end{pmatrix} - B\begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{xt} \end{pmatrix}; \quad B = \begin{pmatrix} D_x\xi & D_x\vartheta & D_x\tau \\ D_y\xi & D_y\vartheta & D_y\tau \\ D_t\xi & D_t\vartheta & D_t\tau \end{pmatrix}$$

this implies that

$$\begin{aligned} \eta^{[xx]} &= D_x\eta^{[x]} - u_{xx}D_x\xi - u_{xy}D_x\vartheta - u_{xt}D_x\tau \\ &= D_x\eta^{[x]} - u_{xx}(\xi_x + u_x\xi_u) - u_{xy}(\vartheta_x + u_x\vartheta_u) - u_{xt}(\tau_x + u_x\tau_u) \end{aligned}$$

From Eq(2.20) we note that

$$\eta^{[x]} = \varphi_x + (\varphi_u - \xi_x)u_x - \vartheta_x u_y - \tau_x u_t - \xi_u u_x^2 - \vartheta_u u_x u_y - \tau_u u_x u_t$$

therefore

$$D_x\eta^{[x]} = (\tfrac{\partial}{\partial x} + u_x \tfrac{\partial}{\partial u} + u_{xx}\tfrac{\partial}{\partial u_x} + u_{xy}\tfrac{\partial}{\partial u_y} + u_{xt}\tfrac{\partial}{\partial u_t})\eta^{[x]}$$

Substituting $\eta^{[x]}$ and simplifying gives.

$$\begin{aligned} D_x\eta^{[x]} = {} & \varphi_{xx} + (2\varphi_{ux} - \xi_{xx})u_x - \vartheta_{xx}u_y - \tau_{xx}u_t + (\varphi_{uu} - 2\xi_{ux})u_x^2 - 2\vartheta_{ux}u_xu_y - 2\tau_{ux}u_xu_t \\ & - \xi_{uu}u_x^3 - \vartheta_{uu}u_x^2u_y - \tau_{uu}u_x^2u_t + (\varphi_u - \xi_x)u_{xx} - \vartheta_x u_{xy} - \tau_x u_{xt} - 2\xi_u u_{xx}u_x \\ & - \vartheta_u u_{xx}u_y - \vartheta_u u_{xy}u_x - \tau_u u_{xx}u_t - \tau_u u_{xt}u_x \end{aligned}$$

implying that

$$\eta^{[xx]} = \varphi_{xx} + (2\varphi_{ux} - \xi_{xx})u_x - \vartheta_{xx}u_y - \tau_{xx}u_t + (\varphi_{uu} - 2\xi_{ux})u_x^2 - 2\vartheta_{ux}u_xu_y$$
$$-2\tau_{ux}u_xu_t - \xi_{uu}u_x^3 - \vartheta_{uu}u_x^2u_y - \tau_{uu}u_x^2u_t + (\varphi_u - 2\xi_x)u_{xx} - 2\vartheta_xu_{xy} - 2\tau_xu_{xt}$$
$$-3\xi_uu_{xx}u_x - \vartheta_uu_{xx}u_y - 2\vartheta_uu_{xy}u_x - \tau_uu_{xx}u_t - 2\tau_uu_{xt}u_x \tag{2.32}$$

To obtain $\eta^{[yy]}$, we follow a similar procedure.

From Eq(2.9)

$$du^*_{y^*} = du_y + \varepsilon d\eta^{[y]} + O(\varepsilon^2)$$
$$= \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \frac{\partial u_y}{\partial t}dt + \varepsilon[\frac{\partial \eta^{[y]}}{\partial x}dx + \frac{\partial \eta^{[y]}}{\partial y}dy + \frac{\partial \eta^{[y]}}{\partial t}dt + \frac{\partial \eta^{[y]}}{\partial u}du + \frac{\partial \eta^{[y]}}{\partial u_x}du_x$$
$$+ \frac{\partial \eta^{[y]}}{\partial u_y}du_y + \frac{\partial \eta^{[y]}}{\partial u_t}du_t] + O(\varepsilon^2)$$
$$= [\frac{\partial u_y}{\partial x} + \varepsilon(\frac{\partial \eta^{[y]}}{\partial x} + \frac{\partial \eta^{[y]}}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \eta^{[y]}}{\partial u_x}\frac{\partial u_x}{\partial x} + \frac{\partial \eta^{[y]}}{\partial u_y}\frac{\partial u_y}{\partial x} + \frac{\partial \eta^{[y]}}{\partial u_t}\frac{\partial u_t}{\partial x})]dx$$
$$+ [\frac{\partial u_y}{\partial y} + \varepsilon(\frac{\partial \eta^{[y]}}{\partial y} + \frac{\partial \eta^{[y]}}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \eta^{[y]}}{\partial u_x}\frac{\partial u_x}{\partial y} + \frac{\partial \eta^{[y]}}{\partial u_y}\frac{\partial u_y}{\partial y} + \frac{\partial \eta^{[y]}}{\partial u_t}\frac{\partial u_t}{\partial y})]dy$$
$$+ [\frac{\partial u_y}{\partial t} + \varepsilon(\frac{\partial \eta^{[y]}}{\partial t} + \frac{\partial \eta^{[y]}}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \eta^{[y]}}{\partial u_x}\frac{\partial u_x}{\partial t} + \frac{\partial \eta^{[y]}}{\partial u_y}\frac{\partial u_y}{\partial t} + \frac{\partial \eta^{[y]}}{\partial u_t}\frac{\partial u_t}{\partial t})]dt + O(\varepsilon^2)$$

which implies that

$$du^*_{y^*} = (\frac{\partial u_y}{\partial x} + \varepsilon D_x\eta^{[y]})dx + (\frac{\partial u_y}{\partial y} + \varepsilon D_y\eta^{[y]})dy + (\frac{\partial u_y}{\partial t} + \varepsilon D_t\eta^{[y]})dt + O(\varepsilon^2) \tag{2.33}$$

Using equations (2.11), (2.12) and (2.13) in the following formula

$$du^*_{y^*} = \frac{\partial u^*_{y^*}}{\partial x^*}dx^* + \frac{\partial u^*_{y^*}}{\partial y^*}dy^* + \frac{\partial u^*_{y^*}}{\partial t^*}dt^*$$

gives

$$du^*_{y^*} = u^*_{x^*y^*}([1 + \varepsilon D_x\xi]dx + \varepsilon[D_y\xi]dy + \varepsilon[D_t\xi]dt)$$
$$+ u^*_{y^*y^*}(\varepsilon[D_x\vartheta]dx + [1 + \varepsilon D_y\vartheta]dy + \varepsilon[D_t\vartheta]dt)$$
$$+ u^*_{y^*t^*}(\varepsilon[D_x\tau]dx + \varepsilon[D_y\tau]dy + [1 + \varepsilon D_t\tau]dt) + O(\varepsilon^2)$$

Using Eq. (2.33) and the independence of dx, dy and dt gives

$$\frac{\partial u_y}{\partial x} + \varepsilon D_x\eta^{[y]} = u^*_{x^*y^*}[1 + \varepsilon D_x\xi] + \varepsilon u^*_{y^*y^*}[D_x\vartheta] + \varepsilon u^*_{y^*t^*}[D_x\tau]$$
$$\frac{\partial u_y}{\partial y} + \varepsilon D_y\eta^{[y]} = \varepsilon u^*_{x^*y^*}[D_y\xi] + u^*_{y^*y^*}[1 + \varepsilon D_y\vartheta] + \varepsilon u^*_{y^*t^*}[D_y\tau]$$
$$\frac{\partial u_y}{\partial t} + \varepsilon D_t\eta^{[y]} = \varepsilon u^*_{x^*y^*}[D_t\xi] + \varepsilon u^*_{y^*y^*}[D_t\vartheta] + u^*_{y^*t^*}[1 + \varepsilon D_t\tau]$$

Next we now express the above in a matrix form

$$\begin{pmatrix} u_{xy} + \varepsilon D_x \eta^{[y]} \\ u_{yy} + \varepsilon D_y \eta^{[y]} \\ u_{yt} + \varepsilon D_t \eta^{[y]} \end{pmatrix} = \begin{pmatrix} 1+\varepsilon D_x \xi & \varepsilon D_x \vartheta & \varepsilon D_x \tau \\ \varepsilon D_y \xi & 1+\varepsilon D_y \vartheta & \varepsilon D_y \tau \\ \varepsilon D_t \xi & \varepsilon D_t \vartheta & 1+\varepsilon D_t \tau \end{pmatrix} \begin{pmatrix} u^*_{x^*y^*} \\ u^*_{y^*y^*} \\ u^*_{y^*t^*} \end{pmatrix} + O(\varepsilon^2)$$

$$= A \begin{pmatrix} u^*_{x^*y^*} \\ u^*_{y^*y^*} \\ u^*_{y^*t^*} \end{pmatrix} + O(\varepsilon^2)$$

$$= (I + \varepsilon B) \begin{pmatrix} u^*_{x^*y^*} \\ u^*_{y^*y^*} \\ u^*_{y^*t^*} \end{pmatrix} + O(\varepsilon^2)$$

this implies that

$$\begin{pmatrix} u^*_{x^*y^*} \\ u^*_{y^*y^*} \\ u^*_{y^*t^*} \end{pmatrix} = (I + \varepsilon B)^{-1} \begin{pmatrix} u_{xy} + \varepsilon D_x \eta^{[y]} \\ u_{yy} + \varepsilon D_y \eta^{[y]} \\ u_{yt} + \varepsilon D_t \eta^{[y]} \end{pmatrix} + O(\varepsilon^2)$$

$$= (I - \varepsilon B) \begin{pmatrix} u_{xy} + \varepsilon D_x \eta^{[y]} \\ u_{yy} + \varepsilon D_y \eta^{[y]} \\ u_{yt} + \varepsilon D_t \eta^{[y]} \end{pmatrix} + O(\varepsilon^2)$$

$$= \begin{pmatrix} u_{xy} \\ u_{yy} \\ u_{yt} \end{pmatrix} + \varepsilon \left[\begin{pmatrix} D_x \eta^{[y]} \\ D_y \eta^{[y]} \\ D_t \eta^{[y]} \end{pmatrix} - B \begin{pmatrix} u_{xy} \\ u_{yy} \\ u_{yt} \end{pmatrix} \right] + O(\varepsilon^2) \tag{2.34}$$

Now comparing the equation (2.25), (2.27), (2.28) and (2.34), we note that

$$\begin{pmatrix} \eta^{[xy]} \\ \eta^{[yy]} \\ \eta^{[yt]} \end{pmatrix} = \begin{pmatrix} D_x \eta^{[y]} \\ D_y \eta^{[y]} \\ D_t \eta^{[y]} \end{pmatrix} - B \begin{pmatrix} u_{xy} \\ u_{yy} \\ u_{yt} \end{pmatrix}; \quad B = \begin{pmatrix} D_x \xi & D_x \vartheta & D_x \tau \\ D_y \xi & D_y \vartheta & D_y \tau \\ D_t \xi & D_t \vartheta & D_t \tau \end{pmatrix}$$

this implies that

$$\begin{aligned} \eta^{[yy]} &= D_y \eta^{[y]} - u_{xy} D_y \xi - u_{yy} D_y \vartheta - u_{yt} D_y \tau \\ &= D_y \eta^{[y]} - u_{xy}(\xi_y + u_y \xi_u) - u_{yy}(\vartheta_y + u_y \vartheta_u) - u_{yt}(\tau_y + u_y \tau_u) \end{aligned}$$

Recall from equation (2.21) that

$$\eta^{[y]} = \varphi_y - \xi_y u_x + (\varphi_u - \vartheta_y) u_y - \tau_y u_t - \xi_u u_x u_y - \vartheta_u u_y^2 - \tau_u u_y u_t$$

Implying that

$$D_y\eta^{[y]} = (\tfrac{\partial}{\partial y} + u_y \tfrac{\partial}{\partial u} + u_{xy} \tfrac{\partial}{\partial u_x} + u_{yy} \tfrac{\partial}{\partial u_y} + u_{yt} \tfrac{\partial}{\partial u_t})\eta^{[y]}$$

Substituting $\eta^{[y]}$ in the above and simplifying gives

$$\begin{aligned} D_y\eta^{[y]} = {} & \varphi_{yy} - \xi_{yy}u_x + (2\varphi_{uy} - \vartheta_{yy})u_y - \tau_{yy}u_t - 2\xi_{uy}u_xu_y + (\varphi_{uu} - 2\vartheta_{uy})u_y^2 - 2\tau_{uy}u_yu_t \\ & - \xi_{uu}u_xu_y^2 - \vartheta_{uu}u_y^3 - \tau_{uu}u_y^2u_t - \xi_yu_{xy} + (\varphi_u - \vartheta_y)u_{yy} - \tau_yu_{yt} - \xi_uu_{yy}u_x \\ & - 2\vartheta_uu_{yy}u_y - \xi_uu_{xy}u_y - \tau_uu_{yy}u_t - \tau_uu_{yt}u_y \end{aligned}$$

Implying that

$$\begin{aligned} \eta^{[yy]} = {} & \varphi_{yy} - \xi_{yy}u_x + (2\varphi_{uy} - \vartheta_{yy})u_y - \tau_{yy}u_t - 2\xi_{uy}u_xu_y + (\varphi_{uu} - 2\vartheta_{uy})u_y^2 - 2\tau_{uy}u_yu_t \\ & - \xi_{uu}u_xu_y^2 - \vartheta_{uu}u_y^3 - \tau_{uu}u_y^2u_t - 2\xi_yu_{xy} + (\varphi_u - 2\vartheta_y)u_{yy} - 2\tau_yu_{yt} - \xi_uu_{yy}u_x \\ & - 3\vartheta_uu_{yy}u_y - 2\xi_uu_{xy}u_y - \tau_uu_{yy}u_t - 2\tau_uu_{yt}u_y \end{aligned} \tag{2.35}$$

Finally we obtain $\eta^{[tt]}$ following the same procedure

From Eq(2.9)

$$\begin{aligned} du^*_{t^*} &= du_t + \varepsilon d\eta^{[t]} + O(\varepsilon^2) \\ &= \tfrac{\partial u_t}{\partial x}dx + \tfrac{\partial u_t}{\partial y}dy + \tfrac{\partial u_t}{\partial t}dt + \varepsilon[\tfrac{\partial \eta^{[t]}}{\partial x}dx + \tfrac{\partial \eta^{[t]}}{\partial y}dy + \tfrac{\partial \eta^{[t]}}{\partial t}dt + \tfrac{\partial \eta^{[t]}}{\partial u}du + \tfrac{\partial \eta^{[t]}}{\partial u_x}du_x \\ &\quad + \tfrac{\partial \eta^{[t]}}{\partial u_y}du_y + \tfrac{\partial \eta^{[t]}}{\partial u_t}du_t] + O(\varepsilon^2) \\ &= [\tfrac{\partial u_t}{\partial x} + \varepsilon(\tfrac{\partial \eta^{[t]}}{\partial x} + \tfrac{\partial \eta^{[t]}}{\partial u}\tfrac{\partial u}{\partial x} + \tfrac{\partial \eta^{[t]}}{\partial u_x}\tfrac{\partial u_x}{\partial x} + \tfrac{\partial \eta^{[t]}}{\partial u_y}\tfrac{\partial u_y}{\partial x} + \tfrac{\partial \eta^{[t]}}{\partial u_t}\tfrac{\partial u_t}{\partial x})]dx \\ &\quad + [\tfrac{\partial u_t}{\partial y} + \varepsilon(\tfrac{\partial \eta^{[t]}}{\partial y} + \tfrac{\partial \eta^{[t]}}{\partial u}\tfrac{\partial u}{\partial y} + \tfrac{\partial \eta^{[t]}}{\partial u_x}\tfrac{\partial u_x}{\partial y} + \tfrac{\partial \eta^{[t]}}{\partial u_y}\tfrac{\partial u_y}{\partial y} + \tfrac{\partial \eta^{[t]}}{\partial u_t}\tfrac{\partial u_t}{\partial y})]dy \\ &\quad + [\tfrac{\partial u_t}{\partial t} + \varepsilon(\tfrac{\partial \eta^{[t]}}{\partial t} + \tfrac{\partial \eta^{[t]}}{\partial u}\tfrac{\partial u}{\partial t} + \tfrac{\partial \eta^{[t]}}{\partial u_x}\tfrac{\partial u_x}{\partial t} + \tfrac{\partial \eta^{[t]}}{\partial u_y}\tfrac{\partial u_y}{\partial t} + \tfrac{\partial \eta^{[t]}}{\partial u_t}\tfrac{\partial u_t}{\partial t})]dt + O(\varepsilon^2) \end{aligned}$$

which implies that

$$du^*_{y^*} = (\tfrac{\partial u_t}{\partial x} + \varepsilon D_x\eta^{[y]})dx + (\tfrac{\partial u_t}{\partial y} + \varepsilon D_y\eta^{[y]})dy + (\tfrac{\partial u_t}{\partial t} + \varepsilon D_t\eta^{[y]})dt + O(\varepsilon^2) \tag{2.36}$$

Using equations (2.11), (2.12) and (2.13) in the following formula

$$du^*_{t^*} = \tfrac{\partial u^*_{t^*}}{\partial x^*}dx^* + \tfrac{\partial u^*_{t^*}}{\partial y^*}dy^* + \tfrac{\partial u^*_{t^*}}{\partial t^*}dt^*$$

gives

$$
\begin{aligned}
du^*_{t^*} &= u^*_{x^*t^*}([1+\varepsilon D_x\xi]dx + \varepsilon[D_y\xi]dy + \varepsilon[D_t\xi]dt) \\
&\quad + u^*_{y^*t^*}(\varepsilon[D_x\vartheta]dx + [1+\varepsilon D_y\vartheta]dy + \varepsilon[D_t\vartheta]dt) \\
&\quad + u^*_{t^*t^*}(\varepsilon[D_x\tau]dx + \varepsilon[D_y\tau]dy + [1+\varepsilon D_t\tau]dt) + O(\varepsilon^2)
\end{aligned}
$$

Using Eq. (2.36) and the independence of dx, dy and dt gives

$$
\begin{aligned}
\tfrac{\partial u_t}{\partial x} + \varepsilon D_x\eta^{[y]} &= u^*_{x^*y^*}[1+\varepsilon D_x\xi] + \varepsilon u^*_{y^*y^*}[D_x\vartheta] + \varepsilon u^*_{y^*t^*}[D_x\tau] \\
\tfrac{\partial u_t}{\partial y} + \varepsilon D_y\eta^{[y]} &= \varepsilon u^*_{x^*y^*}[D_y\xi] + u^*_{y^*y^*}[1+\varepsilon D_y\vartheta] + \varepsilon u^*_{y^*t^*}[D_y\tau] \\
\tfrac{\partial u_t}{\partial t} + \varepsilon D_t\eta^{[y]} &= \varepsilon u^*_{x^*y^*}[D_t\xi] + \varepsilon u^*_{y^*y^*}[D_t\vartheta] + u^*_{y^*t^*}[1+\varepsilon D_t\tau]
\end{aligned}
$$

Next we now express the above in a matrix form

$$
\begin{aligned}
\begin{pmatrix} u_{xt} + \varepsilon D_x\eta^{[t]} \\ u_{yt} + \varepsilon D_y\eta^{[t]} \\ u_{tt} + \varepsilon D_t\eta^{[t]} \end{pmatrix} &= \begin{pmatrix} 1+\varepsilon D_x\xi & \varepsilon D_x\vartheta & \varepsilon D_x\tau \\ \varepsilon D_y\xi & 1+\varepsilon D_y\vartheta & \varepsilon D_y\tau \\ \varepsilon D_t\xi & \varepsilon D_t\vartheta & 1+\varepsilon D_t\tau \end{pmatrix} \begin{pmatrix} u^*_{x^*t^*} \\ u^*_{y^*t^*} \\ u^*_{t^*t^*} \end{pmatrix} + O(\varepsilon^2) \\
&= A \begin{pmatrix} u^*_{x^*t^*} \\ u^*_{y^*t^*} \\ u^*_{t^*t^*} \end{pmatrix} + O(\varepsilon^2) \\
&= (I + \varepsilon B) \begin{pmatrix} u^*_{x^*t^*} \\ u^*_{y^*t^*} \\ u^*_{t^*t^*} \end{pmatrix} + O(\varepsilon^2)
\end{aligned}
$$

this implies that

$$
\begin{aligned}
\begin{pmatrix} u^*_{x^*t^*} \\ u^*_{y^*t^*} \\ u^*_{t^*t^*} \end{pmatrix} &= (I + \varepsilon B)^{-1} \begin{pmatrix} u_{xt} + \varepsilon D_x\eta^{[t]} \\ u_{yt} + \varepsilon D_y\eta^{[t]} \\ u_{tt} + \varepsilon D_t\eta^{[t]} \end{pmatrix} + O(\varepsilon^2) \\
&= (I - \varepsilon B) \begin{pmatrix} u_{xy} + \varepsilon D_x\eta^{[t]} \\ u_{yy} + \varepsilon D_y\eta^{[t]} \\ u_{yt} + \varepsilon D_t\eta^{[t]} \end{pmatrix} + O(\varepsilon^2) \\
&= \begin{pmatrix} u_{xt} \\ u_{yt} \\ u_{tt} \end{pmatrix} + \varepsilon \left[\begin{pmatrix} D_x\eta^{[t]} \\ D_y\eta^{[t]} \\ D_t\eta^{[t]} \end{pmatrix} - B \begin{pmatrix} u_{xt} \\ u_{yt} \\ u_{tt} \end{pmatrix} \right] + O(\varepsilon^2) \qquad (2.37)
\end{aligned}
$$

Now comparing the equation (2.26), (2.28), (2.29) and (2.37), we note that

$$\begin{pmatrix} \eta^{[xt]} \\ \eta^{[yt]} \\ \eta^{[tt]} \end{pmatrix} = \begin{pmatrix} D_x\eta^{[t]} \\ D_y\eta^{[t]} \\ D_t\eta^{[t]} \end{pmatrix} - B\begin{pmatrix} u_{xt} \\ u_{yt} \\ u_{tt} \end{pmatrix}; \quad B = \begin{pmatrix} D_x\xi & D_x\vartheta & D_x\tau \\ D_y\xi & D_y\vartheta & D_y\tau \\ D_t\xi & D_t\vartheta & D_t\tau \end{pmatrix}$$

this implies that

$$\begin{aligned}\eta^{[tt]} &= D_t\eta^{[t]} - u_{xt}D_t\xi - u_{yt}D_t\vartheta - u_{tt}D_t\tau \\ &= D_t\eta^{[t]} - u_{xt}(\xi_t + u_t\xi_u) - u_{yt}(\vartheta_t + u_t\vartheta_u) - u_{tt}(\tau_t + u_t\tau_u)\end{aligned}$$

Recall from equation (2.22) that

$$\eta^{[t]} = \varphi_t - \xi_t u_x - \vartheta_t u_y + (\varphi_u - \tau_t)u_t - \xi_u u_x u_t - \vartheta_u u_y u_t - \tau_u u_t^2$$

Implying that

$$D_t\eta^{[t]} = (\tfrac{\partial}{\partial t} + u_t \tfrac{\partial}{\partial u} + u_{xt} \tfrac{\partial}{\partial u_x} + u_{yt} \tfrac{\partial}{\partial u_y} + u_{tt} \tfrac{\partial}{\partial u_t})\eta^{[t]}$$

Substituting $\eta^{[t]}$ in the above and simplifying gives

$$\begin{aligned}D_t\eta^{[t]} = {}& \varphi_{tt} - \xi_{tt}u_x - \vartheta_{tt}u_y + (2\varphi_{ut} - \tau_{tt})u_t - 2\xi_{ut}u_x u_t - 2\vartheta_{ut}u_y u_t + (\varphi_{uu} - 2\tau_{ut})u_t^2 \\ & - \xi_{uu}u_x u_t^2 - \vartheta_{uu}u_y u_t^2 - \tau_{uu}u_t^3 - \xi_t u_{xt} - \vartheta_t u_{yt} + (\varphi_u - \tau_t)u_{tt} - \xi_u u_{xt}u_t \\ & - \vartheta_u u_{yt}u_t - \xi_u u_{tt}u_x - \vartheta_u u_{tt}u_y - 2\tau_u u_{tt}u_t\end{aligned}$$

Implying that

$$\begin{aligned}\eta^{[tt]} = {}& \varphi_{tt} - \xi_{tt}u_x - \vartheta_{tt}u_y + (2\varphi_{ut} - \tau_{tt})u_t - 2\xi_{ut}u_x u_t - 2\vartheta_{ut}u_y u_t + (\varphi_{uu} - 2\tau_{ut})u_t^2 \\ & - \xi_{uu}u_x u_t^2 - \vartheta_{uu}u_y u_t^2 - \tau_{uu}u_t^3 - 2\xi_t u_{xt} - 2\vartheta_t u_{yt} + (\varphi_u - 2\tau_t)u_{tt} - 2\xi_u u_{xt}u_t \\ & - 2\vartheta_u u_{yt}u_t - \xi_u u_{tt}u_x - \vartheta_u u_{tt}u_y - 3\tau_u u_{tt}u_t\end{aligned} \tag{2.38}$$

2.2. Procedure of finding symmetries of PDEs

Now that we have an idea on how to prolong our symmetry generator in order to obtain the required information on how derivatives u are transformed, we now give an overview of the systematic procedure of obtaining symmetries of a PDE.

Consider a PDE

$$F(x, y, t, u, u_x, u_y, u_t, \ldots, \eth^n u) \tag{2.39}$$

where $\eth^n u$ denotes the n^{th} partial derivatives of u with respect to x, y and t

$$\eth^1 u = (u_x, u_y, u_t)$$
$$\eth^2 u = (u_{xx}, u_{xy}, u_{xt}, u_{yy}, u_{yt}, u_{tt})$$
$$\eth^n u = (\tfrac{\partial^n u}{\partial x^n}, \ldots, \tfrac{\partial^n u}{\partial x^{n-i} \partial y^i}, \ldots, \tfrac{\partial^n u}{\partial y^n})$$

Invariance Criterion

Let $X^{[n]}$ denote the n^{th} prolongation of the symmetry operator (2.3), the equation

$$X^{[n]}F|_{F=0} = 0$$

is called the invariance criterion for the symmetries of the PDE(2.39).

The invariance criterion generally leads us to an over determined system of linear PDEs in $\xi(x,y,t,u)$, $\vartheta(x,y,t,u)$, $\tau(x,y,t,u)$ and $\varphi(x,y,t,u)$. This system is often known as a system of *determining equations* and its solution is set of all possible infinitesimals $\xi(x,y,t,u)$, $\vartheta(x,y,t,u)$, $\tau(x,y,t,u)$ and $\varphi(x,y,t,u)$ that satisfy the invariance condition. In summary the steps involved in finding symmetries of the PDE(2.39) are as follows.

1) Find the n^{th} prolongation $X^{[n]}$
2) Apply the prolongation $X^{[n]}$ to F to obtain $X^{[n]}F|_{F=0} = 0$
3) Obtain the system of determining equations from step 2 by comparing coefficients of derivatives of u
4) Simplify and solve the system of determining equations.

As an example, we consider our research problem and we illustrate steps 1-3. This also helps us avoid unnecessary repetitions as the system of determining equations for equations (1.1) and (1.2) are determined at once and will be used throughout our work.

2.2.1. Determining equations of heat equation on surfaces of revolution.

In this section we present the system of determining equations which will be solved to obtain the symmetry algebra of the heat equation on different surfaces of revolution.

Consider the heat equation on a surface of revolution i.e. Eq(1.1) below.

$$u_t = f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy}$$

As already discussed the symmetry operator (2.3) of the Eq(1.1) is of the form

$$X = \xi(x,y,t,u)\frac{\partial}{\partial x} + \vartheta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \varphi(x,y,t,u)\frac{\partial}{\partial u}$$

For the Eq(1.1) we need up to the second prolongation containing information on how x, u_x, u_t, u_{xx} and u_{yy} are transformed and this is given by

$$X^{[2]} = \xi\frac{\partial}{\partial x} + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[t]}\frac{\partial}{\partial u_t} + \eta^{[xx]}\frac{\partial}{\partial u_{xx}} + \eta^{[yy]}\frac{\partial}{\partial u_{yy}}$$

Let

$$F = F(x,y,t,u,u_x,u_t,u_{xx},u_{yy}) = u_t - f'(x)u_x - u_{xx} - e^{-2f(x)}u_{yy}.$$

The invariance criteria give.

$$X^{[2]}F\Big|_{F=0} = 0$$

$$= \xi\frac{\partial}{\partial x}(F) + \eta^{[x]}\frac{\partial}{\partial u_x}(F) + \eta^{[t]}\frac{\partial}{\partial u_t}(F) + \eta^{[xx]}\frac{\partial}{\partial u_{xx}}(F) + \eta^{[yy]}\frac{\partial}{\partial u_{yy}}(F) = 0$$

$$= -\xi f_{xx}u_x + 2\xi f_x e^{-2f}u_{yy} - \eta^{[x]}f_x + \eta^{[t]} - \eta^{[xx]} - \eta^{[yy]}e^{-2f} = 0$$

This implies that

$$-\xi f_{xx}u_x + 2\xi f_x e^{-2f}u_{yy} + \eta^{[t]} = \eta^{[x]}f_x + \eta^{[xx]} + \eta^{[yy]}e^{-2f}$$

Next we compare the coefficients of the derivatives of u and this gives.

$$u_0 : f_x\varphi_x + \varphi_{xx} + e^{-2f}\varphi_{yy} - \varphi_t = 0$$

$$u_x : \xi f_{xx} + \xi_t + f_x\varphi_u - f_x\xi_x + 2\varphi_{xu} - \xi_{xx} - e^{-2f}\xi_{yy} = 0$$

$$u_y : \vartheta_t - f_x\vartheta_x - \vartheta_{xx} - e^{-2f}\vartheta_{yy} + 2e^{-2f}\varphi_{uy} = 0$$

$$u_t : \varphi_u - \tau_t + f_x \tau_x + \tau_{xx} + e^{-2f} \tau_{yy} = 0$$

$$u_{xx} : \varphi_u - 2\xi_x = 0$$

$$u_{xy} : \vartheta_x + e^{-2f} \xi_y = 0$$

$$u_{xt} : \tau_x = 0$$

$$u_{yy} : 2\xi f_x - \varphi_u + 2\vartheta_y = 0$$

$$u_{yt} : \tau_y = 0$$

$$u_x^2 : \varphi_{uu} - \xi_u - 2\xi_{xu} = 0$$

$$u_x u_y : \vartheta_u f_x + 2\vartheta_{ux} + 2e^{-2f} \xi_{uy} = 0$$

$$u_x u_t : \tau_u f_x + 2\tau_{ux} - \xi_u = 0$$

$$u_y^2 : \varphi_{uu} - 2\vartheta_{uy} = 0$$

$$u_y u_t : \vartheta_u - 2\tau_{uy} e^{-2f} = 0$$

$$u_t^2, u_{xx} u_t, u_{xt} u_x, u_{yy} u_t, u_{yt} u_y : \tau_u = 0$$

$$u_{xx} u_x, u_{yy} u_x, u_{xy} u_y : \xi_u = 0$$

$$u_{xx} u_y, u_{xy} u_x, u_{yy} u_y : \vartheta_u = 0$$

$$u_x^3, u_x u_y^2 : \xi_{uu} = 0$$

$$u_x^2 u_y, u_y^3 : \vartheta_{uu} = 0$$

$$u_x^2 u_t, u_t u_y^2 : \tau_{uu} = 0$$

Assuming a lexicographic ordering as $\varphi > \tau > \vartheta > \xi > f$ and $x > y > t > u$

$$e_0 : \xi_u = 0$$

$$e_1 : \vartheta_u = 0$$

$$e_2 : \tau_u = 0$$

$$e_3 : \tau_y = 0$$

$$e_4 : \tau_x = 0$$

$$e_5 : \varphi_{uu} = 0$$

$$e_6 : 2\xi_x - \tau_t = 0$$

$$e_7 : \vartheta_x + e^{-2f}\xi_y = 0$$
$$e_8 : 2\xi f_x - \tau_t + 2\vartheta_y = 0$$
$$e_9 : f_x\varphi_x + \varphi_{xx} + e^{-2f}\varphi_{yy} - \varphi_t = 0$$
$$e_{10} : \vartheta_t - f_x\vartheta_x - \vartheta_{xx} - e^{-2f}\vartheta_{yy} + 2e^{-2f}\varphi_{uy} = 0$$
$$e_{11} : \xi f_{xx} + \xi_t + f_x\tau_t - f_x\xi_x + 2\varphi_{xu} - \xi_{xx} - e^{-2f}\xi_{yy} = 0$$

2.2.2. Determining equations of wave equation on surfaces of revolution.

In this section we give a system of determining equations which will be solved to obtain the symmetry algebra of the wave equation on different surfaces of revolution.

Consider the wave equation on surface of revolution i.e. Eq(1.2) below.

$$u_{tt} = f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy}$$

As already discussed the symmetry operator (2.3) of the Eq(1.2) is of the form

$$X = \xi(x,y,t,u)\frac{\partial}{\partial x} + \vartheta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \varphi(x,y,t,u)\frac{\partial}{\partial u}$$

For the Eq(1.2) we need up to the second prolongation containing information on how x, u_x, u_{xx}, u_{yy} and u_{tt} are transformed and this is given by

$$X^{[2]} = \xi\frac{\partial}{\partial x} + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[xx]}\frac{\partial}{\partial u_{xx}} + \eta^{[yy]}\frac{\partial}{\partial u_{yy}} + \eta^{[tt]}\frac{\partial}{\partial u_{tt}}$$

Let

$$F = F(x,y,t,u,u_x,u_{xx},u_{yy},u_{tt}) = u_{tt} - f'(x)u_x - u_{xx} - e^{-2f(x)}u_{yy}.$$

The invariance criteria give.

$$X^{[2]}F\Big|_{F=0} = 0$$
$$= \xi\frac{\partial}{\partial x}(F) + \eta^{[x]}\frac{\partial}{\partial u_x}(F) + \eta^{[xx]}\frac{\partial}{\partial u_{xx}}(F) + \eta^{[yy]}\frac{\partial}{\partial u_{yy}}(F) + \eta^{[tt]}\frac{\partial}{\partial u_{tt}}(F) = 0$$
$$= -\xi f_{xx}u_x + 2\xi f_x e^{-2f}u_{yy} - \eta^{[x]}f_x - \eta^{[xx]} - e^{-2f}\eta^{[yy]} + \eta^{[tt]} = 0$$

This implies that

$$-\xi f_{xx}u_x + 2\xi f_x e^{-2f}u_{yy} + \eta^{[tt]} = \eta^{[x]}f_x + \eta^{[xx]} + \eta^{[yy]}e^{-2f}$$

Next we compare the coefficients of the derivatives of u and this gives.

$$u_0 : \varphi_{tt} = f_x\varphi_x + \varphi_{xx} + e^{-2f}\varphi_{yy}$$

$$u_x : -\xi f_{xx} - \xi_{tt} = f_x\varphi_u + 2\varphi_{xu} - \xi_{xx} - e^{-2f(x)}\xi_{yy} - f_x\xi_x$$

$$u_y : -\vartheta_{tt} = -f_x\vartheta_x - \vartheta_{xx} + 2e^{-2f}\varphi_{uy} - e^{-2f}\vartheta_{yy}$$

$$u_t : \tau_{tt} = \tau_x f_x + \tau_{xx} + e^{-2f}\tau_{yy} + 2\varphi_{ut}$$

$$u_{xx} : \varphi_u - 2\xi_x = 0$$

$$u_{xy} : 0 = -2\vartheta_x - 2e^{-2f}\xi_y$$

$$u_{xt} : \tau_x - \xi_t = 0$$

$$u_{yy} : 2\xi f_x e^{-2f} = e^{-2f}(\varphi_u - 2\vartheta_y)$$

$$u_{yt} : \vartheta_t - e^{-2f(x)}\tau_y = 0$$

$$u_{tt} : \varphi_u - 2\tau_t = 0$$

$$u_x^2 : -f_x\xi_u \quad - 2\xi_{xu} + \varphi_{uu} = 0$$

$$u_x u_y : -f_x(\vartheta_u) - 2\vartheta_{ux} - 2e^{-2f}\xi_{uy} = 0$$

$$u_x u_t : 2\xi_{ut} - f_x\tau_u - 2\tau_{ux} = 0$$

$$u_y^2 : \varphi_{uu} - 2\vartheta_{uy} = 0$$

$$u_y u_t : \vartheta_{tu} - e^{-2f}\tau_{uy} = 0$$

$$u_t^2 : \varphi_{uu} - 2\tau_{ut} = 0$$

$$u_x^3, u_y^2u_x, u_t^2u_x : \xi_{uu} = 0$$

$$u_x^2u_y, u_y^3, u_t^2u_y : \vartheta_{uu} = 0$$

$$u_t^3, u_y^2u_t, u_x^2u_t : \tau_{uu} = 0$$

$$u_{tt}u_t, u_{yy}u_t, u_{yt}u_y, u_{xx}u_t, u_{xt}u_x : \tau_u = 0$$

$$u_{tt}u_x, u_{yy}u_x, u_{xx}u_x, u_{xy}u_y, u_{xt}u_t : \xi_u = 0$$

$$u_{tt}u_y, u_{yy}u_y, u_{yt}u_t, u_{xx}u_y, u_{xy}u_x : \vartheta_u = 0$$

Assuming a lexicographic ordering as $\varphi > \tau > \vartheta > \xi > f$ and $x > y > t > u$.

$$e_1 : \xi_u = 0$$

$$e_2 : \vartheta_u = 0$$

$$e_3 : \tau_u = 0$$

$$e_4 : \varphi_{uu} = 0$$

$$e_5 : \xi_t - \tau_x = 0$$

$$e_6 : \xi_x - \tau_t = 0$$

$$e_7 : \vartheta_t - e^{-2f}\tau_y = 0$$

$$e_8 : \vartheta_x + e^{-2f}\xi_y = 0$$

$$e_9 : \xi f_x - \xi_x + \vartheta_y = 0$$

$$e_{10} : \varphi_{tt} - f_x\varphi_x - \varphi_{xx} - e^{-2f}\varphi_{yy} = 0$$

$$e_{11} : \tau_x f_x + e^{-2f}\tau_{yy} + 2\varphi_{ut} = 0$$

$$e_{12} : \xi f_{xx} + f_x\xi_x + 2\varphi_{xu} - e^{-2f}\xi_{yy} = 0$$

$$e_{13} : \vartheta_{tt} - \vartheta_{xx} - f_x\vartheta_x + 2e^{-2f}\varphi_{uy} - e^{-2f}\vartheta_{yy} = 0$$

In chapters 3, 4 and 5 we shall directly apply the results from this chapter without going back into the details. For different determining functions, we solve these systems of determining equations obtained in sections 2.2.1 and 2.2.2 to derive the corresponding symmetry algebra for heat and wave equations.

Chapter 3

Symmetries of heat and wave equations on flat surfaces of revolution

Now that we have an idea from chapter 1, of what a surface of revolution is in our perspective and how to write the heat and the wave equations on it. The next task is to carry out a symmetry classification and analysis of the heat and the wave equations on surfaces of revolution.

This chapter is focused on investigation of symmetries of heat and wave equations on the flat surfaces of revolution whereas the symmetry of the heat and wave equations on the general surface of revolution will be carried out respectively in subsequent chapters.

3.1 Flat surfaces of revolution

Consider a surface of revolution $X(x,y)=(v(x),w(x)\cos y,w(x)\sin y)$ generated a unit speed curve $\alpha(x)=(v(x),w(x))$. If for such surface the Gaussian curvature $K=0$, then it is a said to be a flat surface of revolution since the Gaussian curvature is an intrinsic measure of curvature.

Next we find all flat the surfaces of revolution.

Since for flat surfaces the curvature vanishes, it follows from theorem 1.1 that

$$w''(x) = 0 \Rightarrow w'(x) \text{ is a constant}$$

$$v'(x)^2 + w'(x)^2 = 1 \Rightarrow v'(x) = \sqrt{1 - w'(x)^2} \text{ is also a constant}$$

implying that

$$\alpha'(x) = (v'(x), w'(x))$$

is a constant vector which can be expressed in the form below

$$\alpha'(x) = (\cos\phi, \sin\phi); \text{ for constant } \phi = [0, 2\pi)$$

Thus

$$\alpha(x) = (x\cos\phi, x\sin\phi) + (a,b); \quad \alpha(0) = (a,b).$$

For $\phi = 0, \pi$

$$\alpha(x) = (a \pm x, b), \text{ respectively.}$$

This implies that the coordinate patch X takes a form

$$X(x,y) = (a \pm x, b\cos y, b\sin y)$$

which turns out to be a cylinder.

For $\phi = 0.5\pi, 1.5\pi$

$$\alpha(x) = (a, b \pm x) \text{ respectively.}$$

This implies that the coordinate patch X takes a form

$$X(x,y) = (a, (b \pm x)\cos y, (b \pm x)\sin y)$$

which turns out to be a plane.

For $\phi \neq 0,\ 0.5\pi,\ \pi,\ 1.5\pi.$

$$\alpha(x) = (a + x\cos\phi, b + x\sin\phi)$$

This implies that the coordinate patch X takes a form

$$X(x,y) = ((a + x\cos\phi), (b + x\sin\phi)\cos y, (b + x\sin\phi)\sin y)$$

which turns out to be a cone.

3.2 Symmetries of the heat equation on flat surfaces revolution.

From section 3.1, we see that there are flat surfaces of revolution. We now proceed by discussing the symmetries of the heat equation on these three flat surfaces of revolution.

3.2.1 Symmetries of the heat equation on a cylinder.

Consider a cylinder parameterized by the coordinate patch X of the form

$$X(x,y) = (a \pm x, b\cos y, b\sin y), \quad b > 0$$

Using the determining function $f(x) = \ln b$, it then follows from Eq(1.1) that the heat equation on a cylinder is of the form

$$u_t = u_{xx} + \frac{1}{b^2} u_{yy}. \tag{3.1}$$

Next we write down the system of determining equations for the equation (3.1) using the system given in section 2.2.1 with $f(x) = \ln b$

$$e_0 : \xi_u = 0$$
$$e_1 : \vartheta_u = 0$$
$$e_2 : \tau_u = 0$$
$$e_3 : \tau_y = 0$$
$$e_4 : \tau_x = 0$$
$$e_5 : \varphi_{uu} = 0$$
$$e_6 : \tau_t - 2\xi_x = 0$$
$$e_7 : \vartheta_x + \frac{1}{b^2}\xi_y = 0$$
$$e_8 : \xi_x - \vartheta_y = 0$$
$$e_9 : \varphi_{xx} + \frac{1}{b^2}\varphi_{yy} - \varphi_t = 0$$
$$e_{10} : \vartheta_t - \vartheta_{xx} - \frac{1}{b^2}\vartheta_{yy} + \frac{2}{b^2}\varphi_{uy} = 0$$
$$e_{11} : \xi_t + 2\varphi_{xu} - \xi_{xx} - \frac{1}{b^2}\xi_{yy} = 0$$

The solution of this system is then as follows.

$$(e_6)_x : \xi_{xx} = 0, \ (e_6)_y : \xi_{xy} = 0.$$
$$(e_7)_x : \vartheta_{xx} = 0, \ (e_8)_y : \vartheta_{yy} = 0$$
$$(e_8)_x : \vartheta_{xy} = 0, \ (e_7)_y : \xi_{yy} = 0$$
$$e_{10} : \vartheta_t + \frac{2}{b^2}\varphi_{uy} = 0$$
$$e_{11} : \xi_t + 2\varphi_{xu} = 0$$
$$(e_7)_t : \vartheta_{tx} + \frac{1}{b^2}\xi_{ty} = 0$$

By $(e_{10})_x$ and $(e_{11})_y$

$$\vartheta_{tx} - \frac{1}{b^2}\xi_{ty} = 0 \Rightarrow \xi_{ty} = 0, \ \vartheta_{tx} = 0$$

By $(e_9)_{ux}$ and $(e_9)_{uy}$ respectively

$$\varphi_{uxt} = 0 \Rightarrow \xi_{tt} = 0 \ \text{ and } \ \varphi_{uyt} = 0 \Rightarrow \vartheta_{tt} = 0.$$

Using $(e_{11})_x$,$(e_{10})_y$ and $(e_8)_t$ in $(e_9)_u$ gives

$$e_{12} : \varphi_{ut} + \xi_{xt} = 0$$

For ξ, $\xi_u = \xi_{yy} = \xi_{tt} = \xi_{ty} = \xi_{xx} = \xi_{xy} = 0$ this implies that

$$\xi = (k_1 t + \tfrac{1}{2}k_2)x + k_4 y + k_5 t + k_6.$$

For τ, $\tau_u = \tau_y = \tau_x = \tau_t - 2\xi_x = 0.$ this implies that

$$\tau = k_1 t^2 + k_2 t + k_3$$

For ϑ, $\vartheta_u = \vartheta_{tt} = 0, \vartheta_x = -\frac{1}{b^2}\xi_y, \ \vartheta_y = \xi_x$

$$\vartheta_x = -\frac{k_4}{b^2} \Rightarrow \vartheta = -\frac{k_4}{b^2}x + g(y,t)$$
$$\vartheta_y = k_1 t + \tfrac{1}{2}k_2 \Rightarrow g_y = k_1 t + \tfrac{1}{2}k_2 \Rightarrow g(y,t) = (k_1 t + \tfrac{1}{2}k_2)y + n(t).$$
$$\vartheta_{tt} = 0 \Rightarrow n(t) = k_7 t + k_8.$$

Implying that

$$\vartheta = -\frac{k_4}{b^2}x + (k_1 t + \tfrac{1}{2}k_2)y + k_7 t + k_8.$$

For φ, $\varphi_{uu} = 0, \ \varphi_{ut} = -\xi_{xt}, \ \varphi_{xu} = -\tfrac{1}{2}\xi_t, \ \varphi_{uy} = -\tfrac{1}{2}b^2\vartheta_t.$

$$\varphi_{ut} = -k_1 \Rightarrow \varphi_u = -k_1 t + p(x,y)$$
$$\varphi_{ux} = -\tfrac{1}{2}(k_1 x + k_5) \Rightarrow p(x,y) = -\tfrac{1}{2}(\tfrac{1}{2}k_1 x + k_5)x + q(y)$$

$$\varphi_{uy} = -\tfrac{1}{2}b^2(k_1 y + k_7) \Rightarrow q(y) = -\tfrac{1}{2}b^2(\tfrac{1}{2}k_1 y + k_7)y + k_9.$$

Implying that

$$\varphi = (-k_1 t - \tfrac{1}{2}(\tfrac{1}{2}k_1 x + k_5)x - \tfrac{1}{2}b^2(\tfrac{1}{2}k_1 y + k_7)y + k_9)u + h(x,y,t).$$

where $h(x,y,t)$ satisfies the Eq(3.1)

Hence the associated symmetry algebra in this case is:

$$X_1 = tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + t^2\frac{\partial}{\partial t} - \tfrac{1}{4}(4t + x^2 + b^2y^2)u\frac{\partial}{\partial u}.$$

$$X_2 = \tfrac{1}{2}x\frac{\partial}{\partial x} + \tfrac{1}{2}y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}$$

$$X_3 = \frac{\partial}{\partial t}$$

$$X_4 = y\frac{\partial}{\partial x} - \frac{x}{b^2}\frac{\partial}{\partial y}$$

$$X_5 = t\frac{\partial}{\partial x} - \tfrac{1}{2}xu\frac{\partial}{\partial u}$$

$$X_6 = \frac{\partial}{\partial x}$$

$$X_7 = t\frac{\partial}{\partial y} - \tfrac{1}{2}b^2yu\frac{\partial}{\partial u}$$

$$X_8 = \frac{\partial}{\partial y}$$

$$X_9 = u\frac{\partial}{\partial u}$$

$$X_h = h(x,y,t)\frac{\partial}{\partial u}.$$

3.2.2 Symmetries of the heat equation on a cone.

Consider a cone with a parameterization of the form

$$X(x,y) = ((a + x\cos\phi),(b + x\sin\phi)\cos y,(b + x\sin\phi)\sin y);\quad \sin\phi > 0$$

This implies in the notation of section 1.2 that the determining function

$$f(x) = \ln(b + x\sin\phi),\quad b + x\sin\phi > 0$$

Hence the heat equation on a cylinder is of the form

$$u_t = \frac{\sin\phi}{b + x\sin\phi} u_x + u_{xx} + \frac{1}{(b + x\sin\phi)^2} u_{yy}. \tag{3.2}$$

For simplicity we let $l(x+d) = x\sin\phi + b$

Next we write down the system of determining equations for the equation (3.2) discussed in section 2.2.1 with $f(x) = \ln l(x+d)$

$$e_0 : \xi_u = 0$$
$$e_1 : \vartheta_u = 0$$
$$e_2 : \tau_u = 0$$
$$e_3 : \tau_y = 0$$
$$e_4 : \tau_x = 0$$
$$e_5 : \varphi_{uu} = 0$$
$$e_6 : \tau_t - 2\xi_x = 0$$
$$e_7 : \vartheta_x + (l(x+d))^{-2}\xi_y = 0$$
$$e_8 : \xi(x+d)^{-1} - \xi_x + \vartheta_y = 0$$
$$e_9 : (x+d)^{-1}\varphi_x + \varphi_{xx} + (l(x+d))^{-2}\varphi_{yy} - \varphi_t = 0$$
$$e_{10} : \vartheta_t - (x+d)^{-1}\vartheta_x - \vartheta_{xx} - (l(x+d))^{-2}\vartheta_{yy} + 2(l(x+d))^{-2}\varphi_{uy} = 0$$
$$e_{11} : -\xi(x+d)^{-2} + \xi_t + \xi_x(x+d)^{-1} + 2\varphi_{xu} - \xi_{xx} - (l(x+d))^{-2}\xi_{yy} = 0$$

The solution of the above this system is then as follows.

$$(e_6)_x : \xi_{xx} = 0,\ (e_6)_y : \xi_{xy} = 0.$$

By $(e_8)_x - (e_7)_y$

$$e_{12} : -\xi(x+d)^{-2} + \xi_x(x+d)^{-1} - (l(x+d))^{-2}\xi_{yy} = 0$$

Using e_7, $(e_7)_x$ and $(e_8)_y$ we obtain

$$e_{10} : \vartheta_t + 2(l(x+d))^{-2}\varphi_{uy} = 0$$

Using $(e_6)_x$ and e_{12} gives

$$e_{11} : \xi_t + 2\varphi_{xu} = 0$$

Substituting $(e_{11})_x$ and $(e_8)_t$ in $(e_9)_u$ gives

$$e_{13} : \varphi_{ut} + \xi_{xt} = 0$$

By $(e_{11})_t$ and $(e_{13})_x$ we note that

$$e_{14} : \xi_{tt} = 0.$$

For τ; $\tau_u = \tau_y = \tau_x = 0$, $(e_6)_{tt} : \tau_{ttt} = 0.$ This implies that

$$\tau = k_1 t^2 + k_2 t + k_3.$$

For ξ; $e_0 : \xi_u = 0$, $e_6 : \xi_x = \frac{1}{2}\tau_t = (k_1 t + \frac{1}{2}k_2)$ implying that

$$\xi = (k_1 t + \tfrac{1}{2}k_2)x + g(y,t).$$

By $(e_{11})_t$ we have

$$\xi_{tt} = 0 \Rightarrow g_{tt} = 0 \Rightarrow g(y,t) = h(y)t + i(y)$$

By e_{12} we note that

$$h_{yy}t + i_{yy} = l^2((dk_1 t - h(y))t + \tfrac{1}{2}dk_2 - i(y))$$

Comparing coefficients of t gives

$$h_{yy} + l^2 h(y) = l^2 dk_1$$

$$2i_{yy} + 2l^2 i(y) = l^2 dk_2$$

Solving the above two equations gives

$$h(y) = k_4 \sin(ly) + k_5 \cos(ly) + dk_1$$

$$i(y) = k_6 \sin(ly) + k_7 \cos(ly) + \tfrac{1}{2}dk_2$$

Therefore

$$\xi = (k_1 t + \tfrac{1}{2}k_2)x + (k_4 \sin(ly) + k_5 \cos(ly) + dk_1)t + (k_6 \sin(ly) + k_7 \cos(ly) + \tfrac{1}{2}dk_2)$$

For ϑ; $\vartheta_u = 0$, $e_7 : l^2(x+d)^2\vartheta_x + \xi_y = 0$, $e_8 : \xi + (x+d)(\vartheta_y - \xi_x) = 0.$

By $(e_{10})_t$ we note that

$$\vartheta_{tt} = 0 \Rightarrow \vartheta = k(x,y)t + c(x,y)$$

By e_7 we observe that

$$l(x+d)^2(k_x t + c_x) + (k_4 \cos(ly) - k_5 \sin(ly))t + (k_6 \cos(ly) - k_7 \sin(ly)) = 0$$

Comparing coefficients of t

$$k_x + \frac{(k_4 \cos(ly) - k_5 \sin(ly))}{l(x+d)^2} = 0 \Rightarrow k(x,y) - \frac{(k_4 \cos(ly) - k_5 \sin(ly))}{l(x+d)} = R(y)$$

$$c_x + \frac{(k_6\cos(ly) - k_7\sin(ly))}{l(x+d)^2} = 0 \Rightarrow c(x,y) - \frac{(k_6\cos(ly) - k_7\sin(ly))}{l(x+d)} = T(y)$$

By e_8 we note that

$$(k_4\sin(ly) + k_5\cos(ly) + dk_1)t + (k_6\sin(ly) + k_7\cos(ly) + \tfrac{1}{2}dk_2) + (x+d)(k_y t + c_y)$$
$$- d(k_1 t + \tfrac{1}{2}k_2) = 0$$

Comparing coefficients of t gives

$$(k_4\sin(ly) + k_5\cos(ly)) + (x+d)k_y = 0$$
$$(k_6\sin(ly) + k_7\cos(ly)) + (x+d)c_y = 0$$

Implying that

$$k_y = -\frac{(k_4\sin(ly) + k_5\cos(ly))}{(x+d)}, \quad c_y = -\frac{(k_6\sin(ly) + k_7\cos(ly))}{(x+d)}$$

thus

$$R_y = T_y = 0 \Rightarrow R(y) = k_0 \text{ and } T(y) = k_8$$

Therefore

$$\vartheta = \left(k_0 + \frac{(k_4\cos(ly) - k_5\sin(ly))}{l(x+d)}\right)t + \left(k_8 + \frac{(k_6\cos(ly) - k_7\sin(ly))}{l(x+d)}\right)$$

By differentiating e_{10} twice with respect to x, we note that $k_0 = 0$. Thus

$$\vartheta = \left(\frac{(k_4\cos(ly) - k_5\sin(ly))}{l(x+d)}\right)t + \left(k_8 + \frac{(k_6\cos(ly) - k_7\sin(ly))}{l(x+d)}\right)$$

For φ; $\varphi_{uu} = 0$, $e_{11} : \varphi_{xu} = -\frac{1}{2}\xi_t$, $(e_9)_u : \varphi_{ut} = -\xi_{tx}$, $(e_{10}) : \varphi_{uy} = -\frac{1}{2}\xi_{ty}(x+d)$

$$\varphi_{ux} = -\tfrac{1}{2}(k_1 x + k_4\sin(ly) + k_5\cos(ly) + dk_1)$$
$$\varphi_{uy} = -\tfrac{1}{2}l(x+d)(k_4\cos(ly) - k_5\sin(ly))$$
$$\varphi_{ut} = -k_1$$
$$\varphi_u = -\tfrac{1}{2}(\tfrac{x}{2}k_1 + k_4\sin(ly) + k_5\cos(ly) + dk_1)x + C(y,t)$$
$$\varphi_{uy} = -\tfrac{1}{2}xl(k_4\cos(ly) - k_5\sin(ly)) + C_y \Rightarrow C_y = -\tfrac{1}{2}ld(k_4\cos(ly) - k_5\sin(ly))$$

$$\varphi_{ut} = C_t = -k_1$$
$$C(y,t) = -k_1 t + D(y) \Rightarrow C_y = D_y$$
$$D(y) = -\tfrac{1}{2}d(k_4\sin(ly) + k_5\cos(ly)) + k_9$$

$$\varphi = -\tfrac{1}{2}\left(\tfrac{x}{2}k_1 + k_4\sin(ly) + k_5\cos(ly) + dk_1\right)x - k_1 t - \tfrac{1}{2}d\left(k_4\sin(ly) + k_5\cos(ly)\right) + k_9)u$$
$$+ h(x,y,t).$$

where $h(x,y,t)$ satisfies the Eq(3.2)

Hence the associated symmetry algebra corresponding to $f(x) = \ln l(x+d)$ is

$$X_1 = t(x+d)\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \tfrac{1}{4}(x^2 + 2dx + 4t)u\frac{\partial}{\partial u}$$

$$X_2 = \tfrac{1}{2}(x+d)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$$

$$X_3 = \frac{\partial}{\partial t}$$

$$X_4 = t\sin(ly)\frac{\partial}{\partial x} + \frac{t\cos(ly)}{l(x+d)}\frac{\partial}{\partial y} - \tfrac{1}{2}\sin(ly)(x+d)u\frac{\partial}{\partial u}$$

$$X_5 = t\cos(ly)\frac{\partial}{\partial x} - \frac{t\sin(ly)}{l(x+d)}\frac{\partial}{\partial y} - \tfrac{1}{2}\cos(ly)(x+d)u\frac{\partial}{\partial u}$$

$$X_6 = \sin(ly)\frac{\partial}{\partial x} + \frac{\cos(ly)}{l(x+d)}\frac{\partial}{\partial y}$$

$$X_7 = \cos(ly)\frac{\partial}{\partial x} - \frac{\sin(ly)}{l(x+d)}\frac{\partial}{\partial y}$$

$$X_8 = \frac{\partial}{\partial y}$$

$$X_9 = u\frac{\partial}{\partial u}$$

$$X_h = h(x,y,t)\frac{\partial}{\partial u}.$$

3.2.3 Symmetries of the heat equation on a plane.

Consider a plane with a parameterization of the of the form

$$X(x,y) = (a, (b \pm x)\cos y, (b \pm x)\sin y); \qquad (b \pm x) > 0$$

Without loss of generality, we consider the determining function

$$f(x) = \ln(b + x)$$

Thus the heat equation on a plane is of the form

$$u_t = \frac{1}{b+x}u_x + u_{xx} + \frac{1}{(b+x)^2}u_{yy}. \tag{3.3}$$

Using the results of the previous analysis in (3.2.2) by setting $l=1,\ b=d$, we observe that associated symmetry algebra corresponding to the equation 3.3 is as follows.

$$X_1 = t(x+b)\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \tfrac{1}{4}(x^2+2bx+4t)u\frac{\partial}{\partial u}$$

$$X_2 = \tfrac{1}{2}(x+b)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$$

$$X_3 = \frac{\partial}{\partial t}$$

$$X_4 = t\sin(y)\frac{\partial}{\partial x} + \frac{t\cos(y)}{(x+b)}\frac{\partial}{\partial y} - \tfrac{1}{2}\sin(y)(x+b)u\frac{\partial}{\partial u}$$

$$X_5 = t\cos(y)\frac{\partial}{\partial x} - \frac{t\sin(y)}{(x+b)}\frac{\partial}{\partial y} - \tfrac{1}{2}\cos(y)(x+b)u\frac{\partial}{\partial u}$$

$$X_6 = \sin(y)\frac{\partial}{\partial x} + \frac{\cos(y)}{(x+b)}\frac{\partial}{\partial y}$$

$$X_7 = \cos(y)\frac{\partial}{\partial x} - \frac{\sin(y)}{(x+b)}\frac{\partial}{\partial y}$$

$$X_8 = \frac{\partial}{\partial y}$$

$$X_9 = u\frac{\partial}{\partial u}$$

$$X_h = q(x,y,t)\frac{\partial}{\partial u}.$$

where $h(x,y,t)$ satisfies the Eq(3.3)

3.3 Symmetries of the wave equation on flat surfaces of revolution.

In previous section, we discussed the symmetry algebras of the heat equation on the three flat surfaces of revolution. In this section we are going to discuss the symmetry algebras of the wave equation on the flat surface of revolution.

3.3.1 Symmetries of the wave equation on a cylinder.

Consider a cylinder parameterized by the coordinate patch X of the form

$$X(x,y) = (a \pm x, b\cos y, b\sin y), \quad b > 0$$

The determining function is given by $f(x) = \ln b$

It then follows immediately from Eq(1.2), that the wave equation on a cylinder is of the form

$$u_{tt} = u_{xx} + \frac{1}{b^2} u_{yy}. \tag{3.4}$$

Next we write down the system of determining equations of the Eq(3.4) in system obtained in section 2.2.2 with determining function $f(x) = \ln b$

$$e_1 : \xi_u = 0$$
$$e_2 : \vartheta_u = 0$$
$$e_3 : \tau_u = 0$$
$$e_4 : \varphi_{uu} = 0$$
$$e_5 : \xi_t - \tau_x = 0$$
$$e_6 : \xi_x - \tau_t = 0$$
$$e_7 : \vartheta_t - b^{-2}\tau_y = 0$$
$$e_8 : \vartheta_x + b^{-2}\xi_y = 0$$
$$e_9 : -\xi_x + \vartheta_y = 0$$
$$e_{10} : \varphi_{tt} - \varphi_{xx} - b^{-2}\varphi_{yy} = 0$$
$$e_{11} : b^{-2}\tau_{yy} + 2\varphi_{ut} = 0$$
$$e_{12} : 2\varphi_{xu} - b^{-2}\xi_{yy} + \xi_{tt} - \xi_{xx} = 0$$
$$e_{13} : \vartheta_{tt} - \vartheta_{xx} + 2b^{-2}\varphi_{uy} - b^{-2}\vartheta_{yy} = 0$$

The solution of the above this system is then as follows.

By $(e_6)_x - (e_5)_t$ and $(e_6)_t - (e_5)_x$ we respectively note that

$$e_{14} : \tau_{xx} - \tau_{tt} = 0$$
$$e_{15} : \xi_{xx} - \xi_{tt} = 0$$

Substituting $(e_7)_t$ $(e_8)_x$ and $(e_9)_y$ in e_{13} reduces it to

$$e_{13} : \xi_{xy} + 2\varphi_{uy} = 0$$

By $(e_7)_x - (e_8)_t$ we note that

$$e_{16} : \xi_{ty} = 0$$

By $(e_7)_y - (e_9)_t$, we note that

$$e_{17} : \xi_{xt} - b^{-2}\tau_{yy} = 0$$

The above reduces e_{11} to

$$e_{11} : \xi_{xt} + \varphi_{ut} = 0$$

By $(e_9)_x - (e_8)_y$ we have

$$e_{18} : \xi_{xx} + b^{-2}\xi_{yy} = 0$$

This reduces e_{12} to

$$e_{12} : \xi_{xx} + 2\varphi_{xu} = 0$$

By $(e_{10})_u$, e_{11}, e_{12}, e_{13}, we note that

$$-\xi_{ttx} + \xi_{xxx} + b^{-2}\xi_{xyy} = 0$$

and by e_{15} we have

$$e_{19} : \xi_{xyy} = \tau_{tyy} = 0$$

For $\tau; \tau_u = 0$, $\tau_{xy} = 0$ by $(e_5)_y$, $\tau_{tt} - b^{-2}\tau_{yy} = 0$ by e_{17}, $\tau_{xx} - \tau_{tt} = 0$ by e_{14}.
By $(e_{17})_t$ and e_{19}, $(e_5)_{yy}$ and $(e_{17})_y$, $(e_5)_y$ and $(e_{17})_x$ we note that

$$\tau_{ttx} = \tau_{ttt} = \tau_{tty} = 0$$

implying that

$$\tau_{tt} = \tau_{xx} = b^{-2}\tau_{yy} = k_1$$

Solving the above with $\tau_{xy} = 0$ gives

$$\tau = \tfrac{1}{2}k_1x^2 + (k_2t + k_3)x + \tfrac{1}{2}k_1b^2y^2 + (k_4t + k_5)b^2y + \tfrac{1}{2}k_1t^2 + k_6t + k_7$$

For $\xi_u = 0$ by e_1, $\xi_t - \tau_x = 0$ by e_5, $\xi_x - \tau_t = 0$ by e_6, $\tau_{xt} + b^{-2}\xi_{yy} = 0$ by e_{18}
By $\xi_t - \tau_x = 0$ we note that

$$\xi_t = k_1x + k_2t + k_3 \Rightarrow \xi = (k_1x + k_3)t + \tfrac{1}{2}k_2t^2 + j(y, x)$$

Using $\xi_x - \tau_t = 0$, we observe that

$$\xi_x = k_2x + k_4b^2y + k_1t + k_6 \Rightarrow j_x = k_2x + k_4b^2y + k_6.$$

implying that

$$\xi = (k_1 x + k_3)t + \tfrac{1}{2}k_2 t^2 + \tfrac{1}{2}k_2 x^2 + (k_4 b^2 y + k_6)x + i(y)$$

and $\tau_{xt} + b^{-2}\xi_{yy} = 0$ implies that

$$i(y) = -\tfrac{1}{2}b^2 k_2 y^2 - k_8 b^2 y + k_9$$

Thus

$$\xi = (k_1 x + k_3)t + \tfrac{1}{2}k_2(t^2 + x^2) + (k_4 b^2 y + k_6)x - \tfrac{1}{2}b^2 k_2 y^2 - k_8 b^2 y + k_9.$$

For ϑ; $\vartheta_u = 0, \vartheta_t - b^{-2}\tau_y = 0$ by e_7, $\vartheta_x + b^{-2}\xi_y = 0$ by e_8, $\xi_x - \vartheta_y = 0$ by e_9.

By $\xi_x - \vartheta_y = 0 = 0$ we note that

$$\vartheta_y = k_1 t + k_2 x + (k_4 b^2 y + k_6) \Rightarrow \vartheta = (k_1 t + k_2 x)y + (\tfrac{1}{2}k_4 b^2 y^2 + k_6 y) + z(x,t)$$

By $\vartheta_t - b^{-2}\tau_y = 0$ we note that

$$\vartheta_t = k_1 y + (k_4 t + k_5) \Rightarrow z_t = (k_4 t + k_5) \Rightarrow z(x,t) = \tfrac{1}{2}k_4 t^2 + k_5 t + r(x)$$

By $\vartheta_x + b^{-2}\xi_y = 0$ we note that

$$\vartheta_x = -k_4 x + k_2 y + k_8 \Rightarrow r_x = -k_4 x + k_8 \Rightarrow r(x) = -\tfrac{1}{2}k_4 x^2 + k_8 x + k_{10}.$$

Therefore

$$\vartheta = (k_1 t + k_2 x)y + (\tfrac{1}{2}k_4 b^2 y^2 + k_6 y) + \tfrac{1}{2}k_4 t^2 + k_5 t - \tfrac{1}{2}k_4 x^2 + k_8 x + k_{10}.$$

For φ; $\varphi_{uu} = 0$, $\xi_{tx} + 2\varphi_{ut} = 0$, by e_{11}, $\xi_{xx} + 2\varphi_{xu} = 0$ by e_{12}, $\xi_{xy} + 2\varphi_{uy} = 0$ by e_{13}

$$\xi_{xx} + 2\varphi_{xu} = \xi_{xy} + 2\varphi_{uy} = \xi_{xy} + 2\varphi_{uy} = 0$$

implies that

$$\xi_x + 2\varphi_u = k$$

Substituting ξ gives

$$k_1 t + k_2 x + k_4 b^2 y + k_6 + 2\varphi_u = k$$

therefore

$$\varphi = (k_{11} - \tfrac{1}{2}(k_1 t + k_2 x + k_4 b^2 y))u + h(x,y,t)$$

with $h(x,y,t)$ satisfying the Eq(3.4).

Thus the associated symmetry algebra for the Eq(3.4) is

$$X_1 = xt\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + \tfrac{1}{2}(x^2 + b^2y^2 + t^2)\frac{\partial}{\partial t} - \tfrac{1}{2}tu\frac{\partial}{\partial u}$$

$$X_2 = \tfrac{1}{2}(x^2 - b^2y^2 + t^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + tx\frac{\partial}{\partial t} - \tfrac{1}{2}xu\frac{\partial}{\partial u}$$

$$X_3 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}$$

$$X_4 = b^2xy\frac{\partial}{\partial x} - \tfrac{1}{2}(x^2 - b^2y^2 - t^2)\frac{\partial}{\partial y} + tb^2y\frac{\partial}{\partial t} - \tfrac{1}{2}b^2yu\frac{\partial}{\partial u}$$

$$X_5 = t\frac{\partial}{\partial y} + b^2y\frac{\partial}{\partial t}$$

$$X_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}$$

$$X_7 = \frac{\partial}{\partial t}$$

$$X_8 = -b^2y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$X_9 = \frac{\partial}{\partial x}$$

$$X_{10} = \frac{\partial}{\partial y}$$

$$X_{11} = u\frac{\partial}{\partial u}$$

$$X_h = h(x,y,t)\frac{\partial}{\partial u}$$

with $h(x,y,t)$ satisfying the Eq(3.4)

3.3.2 Symmetries of the wave equation on a cone.

Consider a cone parameterized by the coordinate patch $\boldsymbol{X}$ of the form

$$X(x,y) = ((a + x\cos\phi), (b + x\sin\phi)\cos y, (b + x\sin\phi)\sin y); \quad \sin\phi > 0$$

The determining function $f(x) = \ln(b + x\sin\phi)$; $(b + x\sin\phi) > 0$

Hence the wave equation on a cone is of the form

$$u_{tt} = \frac{\sin\phi}{b + x\sin\phi}u_x + u_{xx} + \frac{1}{(b + x\sin\phi)^2}u_{yy} \qquad (3.5)$$

For simplicity we let

$$x\sin\phi + b = l(x+d)$$

We now write down the system of determining equations derived in section 2.2.2 with determining function $f(x) = \ln l(x+d)$

$$e_1 : \xi_u = 0$$
$$e_2 : \vartheta_u = 0$$
$$e_3 : \tau_u = 0$$
$$e_4 : \varphi_{uu} = 0$$
$$e_5 : \xi_t - \tau_x = 0$$
$$e_6 : \xi_x - \tau_t = 0$$
$$e_7 : l^2(x+d)^2\vartheta_t - \tau_y = 0$$
$$e_8 : l^2(x+d)^2\vartheta_x + \xi_y = 0$$
$$e_9 : \xi - (x+d)(\xi_x - \vartheta_y) = 0$$
$$e_{10} : l^2(x+d)^2(\varphi_{tt} - \varphi_{xx}) - l^2(x+d)\varphi_x - \varphi_{yy} = 0$$
$$e_{11} : \tau_x(x+d)^{-1} + (l(x+d))^{-2}\tau_{yy} + 2\varphi_{ut} = 0$$
$$e_{12} : -(x+d)^{-2}\xi + (x+d)^{-1}\xi_x + 2\varphi_{xu} - (l(x+d))^{-2}\xi_{yy} + \xi_{tt} - \xi_{xx} = 0$$
$$e_{13} : \vartheta_{tt} - \vartheta_{xx} - (x+d)^{-1}\vartheta_x + (2\varphi_{uy} - \vartheta_{yy})(l(x+d))^{-2} = 0$$

The solution of the above this system is then as follows.

By $(e_6)_x - (e_5)_t$ and $(e_6)_t - (e_5)_x$ we obtain the following

$$e_{14} : \tau_{xx} - \tau_{tt} = 0$$
$$e_{15} : \xi_{xx} - \xi_{tt} = 0$$

Using the substitutions $(e_7)_t$, $(e_8)_x$ and $(e_9)_y$ in e_{13} reduces it to

$$e_{13} : \xi_{xy} + 2\varphi_{uy} = 0$$

By $(e_7)_x - (e_8)_t$ we obtain

$$e_{16} : (x+d)^{-1}\tau_y - \xi_{ty} = 0$$

By $(e_7)_y - (e_9)_t$ we obtain

$$e_{17} : \xi_{tx} - (l(x+d))^{-2}\tau_{yy} - (x+d)^{-1}\xi_t = 0$$

This reduces e_{11} to

By $(e_9)_x - (e_8)_y$ we obtain

$$e_{18} : (x+d)^{-1}\xi_x - \xi(x+d)^{-2} - \xi_{xx} - (l(x+d))^{-2}\xi_{yy} = 0$$

This reduces e_{12} to

$$e_{12} : \xi_{xx} + 2\varphi_{xu} = 0$$

Using $(e_{10})_u$, e_{11}, e_{12} and e_{13}, we obtain

$$e_{19} : (x+d)^{-1}\xi_{xx} + (l(x+d))^{-2}\xi_{xyy} = 0$$

Differentiating e_{17} with respect to t, e_6 twice with respect to y and using e_{19}, we obtain

$$e_{20} : \xi_{ttx} = 0$$

and by $(e_{15})_x$ we note that

$$e_{20} : \xi_{xxx} = 0$$

For τ; By e_3, e_{14}, e_{16}, e_{17} and $(e_6)_{tt}$ respectively

$$\tau_u = 0,\ \tau_{xx} - \tau_{tt} = 0,\ \tau_y - (x+d)\tau_{xy} = 0,\ \tau_{yy} = l^2(x+d)^2\tau_{xx} - l^2(x+d)\tau_x \text{ and}$$

$$\tau_{ttt} = \tau_{xxt} = 0.$$

Differentiating e_{16}, e_{14}, e_{19} and e_{14} with respect to x, y, t and x respectively gives

$$\tau_{xxy} = 0,\ \tau_{tty} = 0,\ \tau_{xxx} = 0 \text{ and } \tau_{ttx} = 0.$$

This implies that

$$\tau = \tfrac{1}{2}k_1 t^2 + (h(y)x + g(y))t + p(x,y)$$

It then follows from e_{14} that

$$p(x,y) = \tfrac{1}{2}k_1 x^2 + i(y)x + j(y)$$

$$\tau = \tfrac{1}{2}k_1 t^2 + (h(y)x + g(y))t + \tfrac{1}{2}k_1 x^2 + i(y)x + j(y)$$

Using e_{17} and comparing the coefficients of t we obtain.

$$(h_{yy} + l^2 h(y))x + g_{yy} + l^2 d h(y) = 0$$

$$(-l^2 dk_1 + i_{yy} + l^2 i(y))x + l^2 d i(y) - l^2 d^2 k_1 + j_{yy} = 0$$

Next we compare the coefficients of x and this gives

$$h_{yy} + l^2 h(y) = 0 \Rightarrow h(y) = k_2 \sin(ly) + k_3 \cos(ly)$$

$$g_{yy} + l^2 d h(y) = 0 \Rightarrow g(y) = dk_2 \sin(ly) + dk_3 \cos(ly) + ay + k_4$$

$$i_{yy} + l^2 i(y) = l^2 d k_1 \Rightarrow i(y) = k_5 \sin(ly) + k_6 \cos(ly) + k_1 d$$
$$l^2 d i(y) + j_{yy} = l^2 d^2 k_1 \Rightarrow j(y) = d k_5 \sin(ly) + d k_6 \cos(ly) + by + k_7$$

Using e_{16} and comparing the coefficients of t we obtain

$g_y = dh_y$ and $j_y = di_y$ and this implies that $a = b = 0$.

Therefore

$$\tau = (x+d)[(tk_2 + k_5)\sin(ly) + (tk_3 + k_6)\cos(ly)] + (\tfrac{1}{2}t^2 + \tfrac{1}{2}x^2 + dx)k_1 + k_4 t + k_7.$$

For ϑ; we have $\vartheta_u = 0$, $l^2(x+d)^2\vartheta_t - \tau_y = 0$ by e_2 and e_7. Similarly by $(e_8)_x$, $(e_9)_x$ and $(e_9)_y$ we respectively note that

$$l^2(x+d)^2\vartheta_{xx} + 2l^2(x+d)\vartheta_x + \tau_{ty} = 0$$
$$(x+d)\vartheta_{xy} - (x+d)\tau_{tx} + \vartheta_y = 0$$
$$l^2(x+d)\vartheta_x + (\tau_{ty} - \vartheta_{yy}) = 0$$

By $l^2(x+d)^2\vartheta_t - \tau_y = 0$ we have

$$\vartheta(x,y,t) = \frac{[(k_2 t + 2k_5)\cos(ly) - (k_3 t + 2k_6)\sin(ly)]t}{2l(x+d)} + p(x,y)$$

Using $l^2(x+d)^2\vartheta_{xx} + 2l^2(x+d)\vartheta_x + \tau_{ty} = 0$ we note that

$$p(x,y) = \frac{-(x^2 + xd + d^2)(k_2\cos(ly) - \sin(ly)k_3) + 2l[(x+d)q(y) - g(y)]}{2l(x+d)}$$

Using $(x+d)\vartheta_{xy} - (x+d)\tau_{tx} + \vartheta_y = 0$ and $l^2(x+d)\vartheta_x + (\tau_{ty} - \vartheta_{yy}) = 0$ we solve for $q(y)$ and $g(y)$ substituting them and simplifying gives

$$\vartheta = \frac{(k_8 - 2k_2dx - k_2x^2 + t^2k_2 + 2tk_5)\cos(ly) - (t^2k_3 + 2k_6t - k_9 - k_3x^2 - 2k_3dx)\sin(ly)}{2l(x+d)} + k_{10}$$

For ξ; By $e_9 : \xi = (x+d)(\tau_t - \vartheta_y)$ we note that

$$\begin{aligned}\xi = {} & \tfrac{1}{2}(-k_3x^2 - 2k_3dx - 2d^2k_3 - t^2k_3 - 2k_6t + k_9)\cos(ly) \\ & + \tfrac{1}{2}(-k_2x^2 - 2k_2dx - 2d^2k_2 - t^2k_2 - k_8 - 2tk_5)\sin(ly) - (k_4 + tk_1)(x+d)\end{aligned}$$

For φ; By e_4, e_{12}, e_{11} and e_{13} respectively

$$\varphi_{uu} = 0,\ \xi_{xx} + 2\varphi_{ux} = 0,\quad \xi_{xt} + 2\varphi_{ut} = 0,\ \xi_{xy} + 2\varphi_{uy} = 0$$

By e_4, we note that

$$\varphi = q(x,y,t)u + q(x,y,t)$$

By e_{12}, e_{11} and e_{13}, we note that

$$\xi_x + 2\varphi_u = c$$

thus

$$\varphi_u = q(x,y,t) = \tfrac{1}{2}(c - \xi_x) = \tfrac{1}{2}(c - k_4 - (x+d)[k_2\sin(ly) + k_3\cos(ly)] - k_1 t)$$

Therefore

$$\varphi = (k_{11} - \tfrac{1}{2}(x+d)[k_2\sin(ly) + k_3\cos(ly)] - \tfrac{1}{2}k_1 t)u + h(x,y,t)$$

Where $h(x,y,t)$ satisfies Eq(3.5)

Thus the symmetry algebra of the Eq(3.5) is spanned by

$$X_1 = t(x+d)\frac{\partial}{\partial x} + (\tfrac{1}{2}t^2 + \tfrac{1}{2}x^2 + dx)\frac{\partial}{\partial t} - \tfrac{1}{2}tu\frac{\partial}{\partial u}$$

$$X_2 = \tfrac{1}{2}(t^2 + x^2 + 2xd + 2d^2)\sin(ly)\frac{\partial}{\partial x} + \frac{(t^2 - x^2 - 2xd)\cos(ly)}{2l(x+d)}\frac{\partial}{\partial y} + (x+d)t\sin(ly)\frac{\partial}{\partial t} - \tfrac{1}{2}(x+d)\sin(ly)u\frac{\partial}{\partial u}$$

$$X_3 = \tfrac{1}{2}(t^2 + 2d^2 + x^2 + 2dx)\cos(ly)\frac{\partial}{\partial x} + \frac{(-t^2 + x^2 + 2dx)\sin(ly)}{2l(x+d)}\frac{\partial}{\partial y} + (x+d)t\cos(ly)\frac{\partial}{\partial t} - \tfrac{1}{2}(x+d)\cos(ly)u\frac{\partial}{\partial u}$$

$$X_4 = (x+d)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$$

$$X_5 = t\sin(ly)\frac{\partial}{\partial x} + \frac{t\cos(ly)}{l(x+d)}\frac{\partial}{\partial y} + (x+d)\sin(ly)\frac{\partial}{\partial t}$$

$$X_6 = t\cos(ly)\frac{\partial}{\partial x} - \frac{t\sin(ly)}{l(x+d)}\frac{\partial}{\partial y} + (x+d)\cos(ly)\frac{\partial}{\partial t}$$

$$X_7 = \frac{\partial}{\partial t}$$

$$X_8 = \sin(ly)\frac{\partial}{\partial x} + \frac{\cos(ly)}{l(x+d)}\frac{\partial}{\partial y}$$

$$X_9 = \cos(ly)\frac{\partial}{\partial x} - \frac{\sin(ly)}{l(x+d)}\frac{\partial}{\partial y}$$

$$X_{10} = \frac{\partial}{\partial y}$$

$$X_{11} = u\frac{\partial}{\partial u}$$

$$X_h = h(x,y,t)\frac{\partial}{\partial u}$$

3.3.3 Symmetries of the wave equation on a plane.

Consider a plane parameterized by the coordinate patch X of the form

$$X(x,y) = (a,(b\pm x)\cos y,(b\pm x)\sin y)$$

Without loss of generality, we consider only

$$w(x) = b + x$$

Therefore the determining function is given by $f(x) = \ln(b+x)$. Hence the heat equation on a plane is of the form

$$u_{tt} = \frac{1}{b+x}u_x + u_{xx} + \frac{1}{(b+x)^2}u_{yy}. \tag{3.6}$$

Using the results of the previous analysis in (3.3.2) by setting $l = 1$, we observe that associated symmetry algebra corresponding to the equation (3.6) is as follows.

$$X_1 = t(x+d)\frac{\partial}{\partial x} + (\tfrac{1}{2}t^2 + \tfrac{1}{2}x^2 + dx)\frac{\partial}{\partial t} - \tfrac{1}{2}tu\frac{\partial}{\partial u}$$

$$X_2 = \tfrac{1}{2}(t^2 + x^2 + 2xd + 2d^2)\sin(y)\frac{\partial}{\partial x} + \frac{(t^2 - x^2 - 2xd)\cos(y)}{2(x+d)}\frac{\partial}{\partial y}$$
$$+ (x+d)t\sin(y)\frac{\partial}{\partial t} - \tfrac{1}{2}(x+d)\sin(y)u\frac{\partial}{\partial u}$$

$$X_3 = \tfrac{1}{2}(t^2 + 2d^2 + x^2 + 2dx)\cos(y)\frac{\partial}{\partial x} + \frac{(-t^2 + x^2 + 2dx)\sin(y)}{2(x+d)}\frac{\partial}{\partial y}$$
$$+ (x+d)t\cos(y)\frac{\partial}{\partial t} - \tfrac{1}{2}(x+d)\cos(y)u\frac{\partial}{\partial u}$$

$$X_4 = (x+d)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$$

$$X_5 = t\sin(y)\frac{\partial}{\partial x} + \frac{t\cos(ly)}{l(x+d)}\frac{\partial}{\partial y} + (x+d)\sin(y)\frac{\partial}{\partial t}$$

$$X_6 = t\cos(ly)\frac{\partial}{\partial x} - \frac{t\sin(y)}{(x+d)}\frac{\partial}{\partial y} + (x+d)\cos(y)\frac{\partial}{\partial t}$$

$$X_7 = \frac{\partial}{\partial t}$$

$$X_8 = \sin(y)\frac{\partial}{\partial x} + \frac{\cos(y)}{(x+d)}\frac{\partial}{\partial y}$$

$$X_9 = \cos(y)\frac{\partial}{\partial x} - \frac{\sin(y)}{(x+d)}\frac{\partial}{\partial y}$$

$$X_{10} = \frac{\partial}{\partial y}$$

$$X_{11} = u\frac{\partial}{\partial u}$$

$$X_h = h(x,y,t)\frac{\partial}{\partial u}$$

With this, we complete our discussion of symmetries of heat and wave equations on flat surfaces of revolution. What about for the non flat surfaces of revolution? The answer to this question in general is discussed in the following chapters.

Chapter 4

Classification of non-flat surfaces of revolution according to symmetries of the heat equation

A surface of revolution is said to be non-flat if its Gaussian curvature is not identically zero. In chapter 3, we explicitly discussed the symmetry analysis of heat equation on flat surfaces of revolution. In this chapter, we extend our discussion to non-flat surfaces of revolution. We shall give a complete classification of non-flat surfaces of revolution according to symmetries of heat equation.

The analysis consists of first finding Lie symmetries of the heat equation on an arbitrary surface of revolution with a determining function $f(x)$ and then determining all other forms of $f(x)$ for which larger symmetry groups exist. Precisely the following result will be proved.

Theorem 4.1

The minimal symmetry algebra of the heat equation

$$u_t = f'(x)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + e^{-2f(x)}\frac{\partial^2 u}{\partial y^2}$$

on a surface of revolution parameterized by is generated by

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = u\frac{\partial}{\partial u}$$

and is obtained for an arbitrary determining function f. The larger symmetry algebra exists in the cases given in the table 1.1.

4.1. Symmetry analysis of heat equation on non-flat surface of revolution

Recall the heat equation Eq(1.1) on any surface of revolution parameterized by the coordinate patch

$$X(x,y) = (v(x),\;\; e^{f(x)}\cos y,\;\; e^{f(x)}\sin y),\; 0 \le y < 2\pi$$

is of the form

$$u_t = f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy}.$$

In chapter 2, we noted that that the symmetry generator of the Lie algebra is of the form

$$X = \xi(x,y,t,u)\frac{\partial}{\partial x} + \vartheta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \varphi(x,y,t,u)\frac{\partial}{\partial u}$$

In section 2.2.1, we denoted the second prolongation by $X^{[2]}$ and using the invariance condition,

$$X^{[2]}(u_t - f'(x)u_x - u_{xx} - e^{-2f(x)}u_{yy})\Big|_{u_t = f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy}} = 0$$

we obtained the following system of twelve determining equations.

$$e_0 : \xi_u = 0$$
$$e_1 : \vartheta_u = 0$$
$$e_2 : \tau_u = 0$$
$$e_3 : \tau_y = 0$$
$$e_4 : \tau_x = 0$$
$$e_5 : \varphi_{uu} = 0$$
$$e_6 : 2\xi_x - \tau_t = 0$$
$$e_7 : \vartheta_x + e^{-2f}\xi_y = 0$$
$$e_8 : 2\xi f_x - \tau_t + 2\vartheta_y = 0$$
$$e_9 : f_x\varphi_x + \varphi_{xx} + e^{-2f}\varphi_{yy} - \varphi_t = 0$$
$$e_{10} : \vartheta_t - f_x\vartheta_x - \vartheta_{xx} - e^{-2f}\vartheta_{yy} + 2e^{-2f}\varphi_{uy} = 0$$

$$e_{11} : \xi f_{xx} + \xi_t + f_x \tau_t - f_x \xi_x + 2\varphi_{xu} - \xi_{xx} - e^{-2f} \xi_{yy} = 0$$

We begin by determining the minimal symmetry algebra for the heat equation on an arbitrary surface of revolution with a determining function $f(x)$ by obtaining a triangulation of the determining equations above.

Differentiating e_6 with respect to x and y respectively gives

$$\xi_{xx} = 0 \text{ and } \xi_{xy} = 0.$$

Using $(e_8)_x - (e_7)_y$ we obtain

$$e_{12} : \xi f_{xx} + \xi_x f_x - e^{-2f} \xi_{yy} = 0$$

By e_7, $(e_7)_x$ and $(e_8)_y$, e_{10} reduces to

$$e_{10} : \vartheta_t + 2e^{-2f} \varphi_{uy} = 0$$

Using e_6 and e_{12} we reduce e_{11} to

$$e_{11} : \xi_t + 2\varphi_{xu} = 0$$

Using $e_{11}, (e_{11})_x$, $(e_8)_t - (e_{10})_y$ and $(e_9)_u$ we obtain

$$e_{13} : \varphi_{ut} + \xi_{xt} = 0$$

Using $(e_{11})_t$ and $(e_{13})_x$ we obtain

$$e_{14} : \xi_{tt} = 0$$

Differentiating e_{12} with respect to x gives

$$e_{15} : 2\xi_x f_{xx} + \xi f_{xxx} + 2 f_x e^{-2f} \xi_{yy} = 0$$

Eliminating ξ using e_{12} and e_{15} gives

$$e_{16} : \xi_x (f_x f_{xxx} - 2 f_{xx}^2) - (f_{xxx} + 2 f_x f_{xx}) e^{-2f} \xi_{yy} = 0$$

Eliminating ξ_x using e_{12} and e_{15} gives

$$e_{17}:\xi(f_x f_{xxx} - 2f_{xx}^2) + 2(f_x^2 + f_{xx})e^{-2f}\xi_{yy} = 0$$

Eliminating $e^{-2f}\xi_{yy}$ using e_{16} and e_{17} gives

$$e_{18}:\xi(f_{xxx} + 2f_x f_{xx}) + 2\xi_x(f_x^2 + f_{xx}) = 0$$

Differentiating e_{18} with respect to y gives

$$e_{19}:(f_{xxx} + 2f_x f_{xx})\xi_y = 0.$$

If we now consider an arbitrary $f(x)$, then by e_{19} $\xi_y = 0$. It then follows immediately from e_{17} that $\xi = 0$. By e_2, e_4, e_3 and e_6 we note that $\tau = k_1$

By e_1, e_7, and e_8 we note that $\vartheta = \vartheta(t)$. By e_5, e_{11}, and e_{13} we have $\varphi_u = p(y)$. Substituting the two results in e_{10} implies $\vartheta = k_2$ and $\varphi = k_3 u + g(x,y,t)$ and this gives the minimal symmetry algebra since we assumed an arbitrary $f(x)$.

$$X_1 = \frac{\partial}{\partial y},\quad X_2 = \frac{\partial}{\partial t},\quad X_3 = u\frac{\partial}{\partial u} \text{ and } X_g = g\frac{\partial}{\partial u}.$$

with $g(x,y,t)$ satisfying equation (1.1)

The commutator table for this symmetry algebra is given in chapter 6 (table 6.1)

To look for the determining functions $f(x)$ which may give larger symmetry algebra, we consider the cases $\xi_y \neq 0$ and $\xi_y = 0$ in sections 4.1.1 and 4.1.2 respectively.

4.1.1. Classification for the case $\xi_y \neq 0$

If $\xi_y \neq 0$, then $\xi \neq 0$. This implies that

By e_{19} we note that

$$f_{xxx} + 2f_x f_{xx} = 0 \tag{4.1}$$

This implies that

$$f_{xx} + f_x^2 = k \tag{4.2}$$

This gives to two possible cases

Case 1. $k \neq 0$, this implies by e_{18} $\xi_x = 0$.

For this case we shall first give a general analysis of the symmetries of the heat equation on surface of revolution whose determining function $f(x)$ satisfies the equations (4.1) and (4.2) with $k \neq 0$.

If $f(x)$ satisfies the equations (4.1) and (4.2) with $k \neq 0$ i.e.

$$2f_x f_{xx} + f_{xxx} = 0 \text{ and } f_x^2 + f_{xx} = k,\ \ k \neq 0,$$

then it follows that the system of the determining equations is given by

$$e_0 : \xi_u = 0$$
$$e_1 : \vartheta_u = 0$$
$$e_2 : \tau_u = 0$$
$$e_3 : \tau_y = 0$$
$$e_4 : \tau_x = 0$$
$$e_5 : \varphi_{uu} = 0$$
$$e_6 : \tau_t = 0$$
$$e_7 : \vartheta_x + e^{-2f}\xi_y = 0$$
$$e_8 : \xi f_x + \vartheta_y = 0$$
$$e_9 : f_x\varphi_x + \varphi_{xx} + e^{-2f}\varphi_{yy} - \varphi_t = 0$$
$$e_{10} : \vartheta_t + 2e^{-2f}\varphi_{uy} = 0$$
$$e_{11} : \xi_t + 2\varphi_{xu} = 0$$
$$e_{12} : \xi f_{xx} - e^{-2f}\xi_{yy} = 0$$
$$e_{13} : \varphi_{ut} = 0$$
$$e_{14} : \xi_{tt} = 0$$

By $e^{2f}((e_7)_t - (e_{10})_x) + (e_{11})_y$ we have

$$e_{20} : \xi_{ty} + 2f_x\varphi_{uy} = 0$$

Similarly $\frac{1}{2}(e_{20})_x - f_x((e_{11})_y - e_{20})$ implies that $(f_{xx} + f_x^2)\varphi_{uy} = 0$ thus $\varphi_{uy} = 0$ Using the above result with $(e_9)_u$, e_{13}, and $(e_{11})_x$ we note that $\varphi_{ux} = 0$ thus $\xi_t = 0$. This implies that $\xi = \xi(y)$.

For ξ, we further note that $\forall f(x);\ 2f_xf_{xx} + f_{xxx} = 0,\ e^{2f}f_{xx}$ is a constant function. It then follows immediately from e_{12} that for some $n \in \mathbb{R}$

For $f_{xx} < 0 \Rightarrow e^{2f}f_{xx} = -n^2 \Rightarrow \xi(y) = k_1\sin(ny) + k_2\cos(ny)$
For $f_{xx} = 0 \Rightarrow e^{2f}f_{xx} = 0 \Rightarrow \xi(y) = k_1y + k_2$
For $f_{xx} > 0 \Rightarrow e^{2f}f_{xx} = n^2 \Rightarrow \xi(y) = k_1e^{ny} + k_2e^{-ny}$

For τ; we observe that $\tau = k_3$

For ϑ; $e_{10} : \vartheta_t = 0,\ e_1 : \vartheta_u = 0\ \ e_7 : \vartheta_x + e^{-2f}\xi_y = 0$, $e_8 : \xi f_x + \vartheta_y = 0$
It follows directly from e_{10} and e_1 that $\vartheta = \vartheta(x,y)$
By e_7 we observe that

$$\vartheta(x,y) + \xi_y\int e^{-2f(x)}dx = p(y)$$

Using e_8, we obtain

$$p_y = \xi_{yy}\int e^{-2f(x)}dx - \xi f_x$$

implying that

$$p(y) = \int\left(\xi_{yy}\int e^{-2f(x)}dx - \xi f_x\right)dy + k_4$$

Therefore

$$\vartheta(x,y) = \int\left(\xi_{yy}\int e^{-2f(x)}dx - \xi f_x\right)dy - \xi_y\int e^{-2f(x)}dx + k_4$$

For φ; $\varphi_{uu} = 0,\ \varphi_{ux} = 0,\ \varphi_{uy} = 0,\ \varphi_{ut} = 0 \Rightarrow \varphi = k_5u + g(x,y,t)$
with $g(x,y,t)$ satisfying equation (1.1)
Next we look at the different solutions of the equation (4.2) with $k \neq 0$.

Case 1.1 $f(x)$ is linear i.e. $f_{xx} = 0$ and $f_x \neq 0$

This implies that $f(x) = m^{-1}x + c$ for some constants $m;\ 0 < |m| < \infty$ and c.

If without loss of generality, we let $c = \ln|m|$, then $w(x) = |m|e^{x/m}$

It then immediately follows from the relationship between $v(x)$ and $w(x)$ that

$$v(x) = \int_0^x \sqrt{1 - e^{2t/m}}\,dt; \quad -\infty < x < 0 \ if \ \ m > 0 \text{ and } 0 \le x < \infty \quad if \ \ m < 0.$$

If we further let $|m| = a$ for some $a > 0$, then curve $\alpha(x)$ takes the form

$$\alpha(x) = \begin{cases} \left(\int_0^x \sqrt{1 - e^{2t/a}}\,dt, \quad a e^{x/a} \right); & -\infty < x < 0 \\ \left(\int_0^x \sqrt{1 - e^{-2t/a}}\,dt, \ \ a e^{-x/a} \right); & 0 \le x < \infty \end{cases}$$

This is a unit speed parameterization of the ***tractrix*** and its corresponding surface of revolution is a ***Pseudosphere*** or ***tractoid*** [15].

The coordinate patch for a ***Pseudosphere*** or ***tractoid*** is of the form

$$X(x,y) = \begin{cases} \left(\int_0^x \sqrt{1 - e^{2t/a}}\,dt, \ \ a e^{x/a} \cos y, \ \ a e^{x/a} \sin y \right); & -\infty < x < 0, \ \ 0 \le y < 2\pi \\ \left(\int_0^x \sqrt{1 - e^{-2t/a}}\,dt, \ \ a e^{-x/a} \cos y, \ \ a e^{-x/a} \sin y \right); & 0 \le x < \infty, \ \ 0 \le y < 2\pi \end{cases}$$

Below is an illustration of a Pseudosphere with $a = 4$ and $-8 \le x \le 8$.

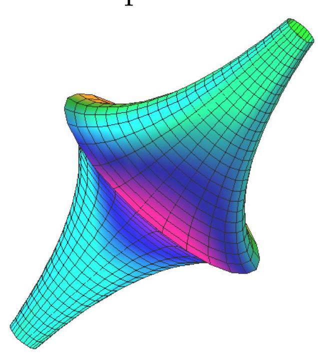


Figure 4.1

The Gaussian curvature for the tractoid in this regard is given by $K = -m^{-2}$ that is negative constant curvature.

Symmetries of the heat equation on a Pseudosphere or tractoid

We now let $m^{-1} = b$. Thus $f(x) = bx + c$ and $w(x) = e^{bx+c}$ implying that

$$\xi = k_1 y + k_2.$$
$$\tau = k_3.$$
$$\vartheta = \frac{k_1}{2b} e^{-2(bx+c)} - \frac{1}{2} by(k_1 y + 2k_2) + k_4$$
$$\varphi = k_5 u + g(x, y, t).$$

Hence the symmetry algebra becomes

$$X_1 = y\frac{\partial}{\partial x} + (\tfrac{1}{2b} e^{-2(bx+c)} - \tfrac{1}{2} by^2)\frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial x} - by\frac{\partial}{\partial y},\ X_3 = \frac{\partial}{\partial t},\ X_4 = u\frac{\partial}{\partial u}$$
$$X_5 = \frac{\partial}{\partial y},\ X_g = g\frac{\partial}{\partial u}$$

Table 4.1:

Commutator table for the symmetry algebra of heat equation on a Pseudosphere

	X_1	X_2	X_3	X_4	X_5
X_1	0	$-bX_1$	0	X_2	0
X_2	bX_1	0	0	$-bX_4$	0
X_3	0	0	0	0	0
X_4	$-X_2$	bX_4	0	0	0
X_5	0	0	0	0	0

Case 1.2 $f(x)$ is non-linear i.e. $f_{xx} \neq 0$

The substitution $f_x = H(x)$ reduces equation (4.2) to

$$H^2 + H_x = k$$

Next we analyze different cases resulting from different values of *k*.

Case1.2.1, $k < 0$, we let $k = -m^2$ for some number $m > 0$

$$H^2 + H_x = -m^2$$

$$H = f_x = -m\tan(mx + mc)$$

This gives

$$f(x) = \ln b|\cos(mx + mc)|; \quad b > 0$$

Let $m = a^{-1}$ and without loss of generality, we consider

- $a > 0,\ c = 0,$ and thus $w(x) = b|\cos(a^{-1}x)|$.
- One case where $w(x) = b\cos(a^{-1}x)$ with x having the following ranges.

$$\begin{aligned} &if\ \ b = a,\ \ then\ \ -\tfrac{1}{2}\pi a < x < \tfrac{1}{2}\pi a; \\ &if\ \ b < a,\ \ then\ \ -\tfrac{1}{2}\pi a < x < \tfrac{1}{2}\pi a; \\ &if\ \ b > a,\ \ then\ \ -a\arcsin\frac{a}{b} < x < a\arcsin\frac{a}{b}. \end{aligned}$$

It then immediately follows from the relationship between $v(x)$ and $w(x)$ that

$$v(x) = \int_0^{x/a} \sqrt{a^2 - b^2\sin^2 t}\,dt.$$

Therefore the coordinate patch of the above surface is given by

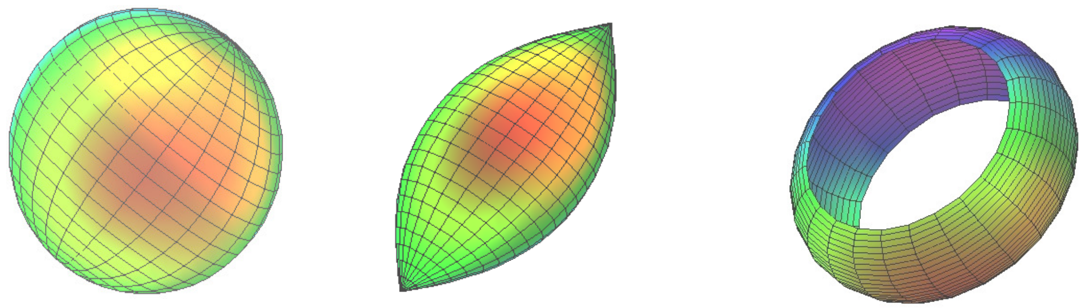
$$X(x,y) = \left(\int_0^{x/a} \sqrt{a^2 - b^2\sin^2 t}\,dt,\ \ b\cos(a^{-1}x)\cos y,\ \ b\cos(a^{-1}x)\sin y\right);\ \ 0 \le y < 2\pi$$

Remarks

Let $S(a,b)$ *be the surface of revolution corresponding to the curve* $\alpha(x) = (v,w)$

- $S(a,a)$ *is an ordinary sphere of radius* a
- $($*Spindle type*$)$ *if* $0 < b < a$ *then* $S(a,b)$ *is a surface of revolution is like a rugby ball with sharp vertices on its axis of revolution.*
- $($*Bulge type*$)$ *if* $0 < a < b,$ *then* $S(a,b)$ *is a barrel - shaped and does not meet axis of revolution.* [15]

Below are the illustrations of examples of the above three cases.



$when\ \ a = b = 1$ $\qquad$ $when\ \ a = 1.5,\ \ b = 1$ $\qquad$ $when\ \ a = 1,\ \ b = 1.5.$

Figure 4.2

We note that these surfaces are of positive Gaussian curvature $K = a^{-2}$

Symmetries of the heat equation on a the surface $S(a,b)$

For $f(x) = \ln b|\cos(mx)|$, $\quad f_{xx} = -m^2 \sec^2 mx \quad$ i.e. $\quad f_{xx} < 0\ \forall x \quad$ and $\quad n^2 = m^2 b^2$
therefore
we have the following infinitesimals.

$$\xi = k_1 \sin(mby) + k_2 \cos(mby)$$

$$\tau = k_3$$

$$\vartheta = k_4 - \frac{1}{b}\tan(mx)(k_1 \sin(mby) - k_2 \cos(mby))$$

$$\varphi = k_5 u + g(x,y,t).$$

Consequently the corresponding symmetry algebra is given by

$$X_1 = \sin(bmy)\frac{\partial}{\partial x} - \frac{1}{b}\tan mx \cos(bmy)\frac{\partial}{\partial y},\ \ X_3 = \frac{\partial}{\partial t},\ \ X_5 = u\frac{\partial}{\partial u}$$

$$X_2 = \cos(bmy)\frac{\partial}{\partial x} + \frac{1}{b}\tan mx \sin(bmy)\frac{\partial}{\partial y},\ \ X_4 = \frac{\partial}{\partial y},\ \ X_g = g\frac{\partial}{\partial u}$$

Table 4.2:

Commutator table for the symmetry algebra of heat equation on the surface described as $S(a,b)$

	X_1	X_2	X_3	X_4	X_5
X_1	0	$-mb^{-1}X_4$	0	bX_2	0
X_2	$mb^{-1}X_4$	0	0	$-bX_1$	0
X_3	0	0	0	0	0
X_4	$-bX_2$	bX_1	0	0	0
X_5	0	0	0	0	0

Case 1.2.2. When $k > 0$ we let $k = r^2,\ r > 0$

$$H^2 + H_x = r^2$$

$$\ln\left|\frac{H+r}{H-r}\right| = 2r(x+p)$$

This gives two possible cases below.

Case 1.2.2.1, when $H >| r |$

$$\ln\left(\frac{H+r}{H-r}\right) = 2r(x+p)$$

$$H = f_x = \frac{r(1+e^{2r(x+p)})}{e^{2r(x+p)}-1}$$

This implies that

$$f(x) = \ln b\,|\sinh(r(x+p))|; \quad b > 0$$

Let $r = a^{-1},\ a \neq 0$, and without loss of generality

- We consider $a > 0,\ p = 0,$ and thus $w(x) = b|\sinh(a^{-1}x)|$ implying that
- We further consider the first case $w(x) = b\sinh(a^{-1}x);\ x > 0.$

It then follows from the relationship between $v(x)$ and $w(x)$ that

$$v(x) = \int_0^{x/a} \sqrt{a^2 - b^2 \cosh^2 t}\,dt;\; b \le a \text{ and } 0 < x \le a\,\text{arcsinh}\left(\frac{\sqrt{b^2 - a^2}}{b}\right)$$

Therefore this surface of revolution is parameterized by the coordinate patch

$$X(x,y) = \left(\int_0^{x/a} \sqrt{a^2 - b^2 \cosh^2 t}\,dt,\;\; b\sinh(a^{-1}x)\cos y,\;\; b\sinh(a^{-1}x)\sin y\right)$$

Below is an example of an illustration such a surface of revolution and it's of a ***conic type.*** [15] *With* $a = 4$ *and* $b = 2$.

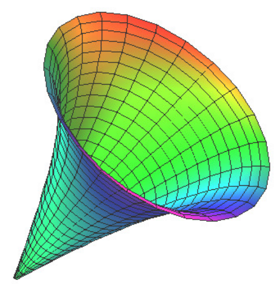


Figure 4.3

It can be shown the Gaussian curvature for this surface is given by $K = -a^{-2}$

Symmetries of heat equation on a conic type surface

For $f(x) = \ln b|\sinh(rx)|$, $f_{xx} = -r^2 \operatorname{sech}^2 rx$ i.e. $f_{xx} < 0\ \forall x$ and $n^2 = r^2 b^2$ therefore we have the following infinitesimals.

$$\xi = k_1 \sin(rby) + k_2 \cos(rby)$$

$$\vartheta = k_4 + \frac{(k_1 \sin(rby) - k_2 \cos(rby))}{b \tanh rx}$$

$$\tau = k_3$$

$$\varphi = k_5 u + g(x,y,t).$$

Thus the corresponding symmetry algebra is given by;

$$X_1 = \sin(bry)\frac{\partial}{\partial x} + \frac{\cos(bry)}{b\tanh(rx)}\frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial t}, \quad X_5 = u\frac{\partial}{\partial u},$$
$$X_2 = \cos(bry)\frac{\partial}{\partial x} - \frac{\sin(bry)}{b\tanh(rx)}\frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial y}, \quad X_g = g\frac{\partial}{\partial u}.$$

Table 4.3:

Commutator table for the symmetry algebra of heat equation on a surface of a conic type

	X_1	X_2	X_3	X_4	X_5
X_1	0	$rb^{-1}X_4$	0	brX_2	0
X_2	$-rb^{-1}X_4$	0	0	$-brX_1$	0
X_3	0	0	0	0	0
X_4	$-brX_2$	brX_1	0	0	
X_5	0	0	0	0	0

Case 1.2.2.2, when $H <| r |$

$$\ln\left(\frac{H+r}{r-H}\right) = 2r(x+p)$$

$$H = f_x = \frac{r(e^{2r(x+p)}-1)}{e^{2r(x+p)}+1}$$

This implies that

$$f(x) = \ln b\cosh(r(x+p)); \quad b > 0$$

Let $r = a^{-1}$, $a \neq 0$, and without loss of generality we consider $p = 0$

It then follows from the relationship between $v(x)$ and $w(x)$ that

$$v(x) = \int_0^{x/a} \sqrt{a^2 - b^2\sinh^2 t}\,dt; \quad -a\,\mathrm{arcsinh}\,\frac{b}{a} < x \le a\,\mathrm{arcsinh}\,\frac{b}{a}$$

Therefore this surface of revolution is parameterized by the coordinate patch

$$X(x,y)=\left(\int_0^{x/a}\sqrt{a^2-b^2\sinh^2 t}\,dt,\ \ b\cosh(a^{-1}x)\cos y,\ \ b\sinh(a^{-1}x)\sin y\right)$$

Below is an illustration such a surface of revolution and it's of a ***Hyperboloid of one sheet type***. [15] With $a = 4$ and $b = 2$

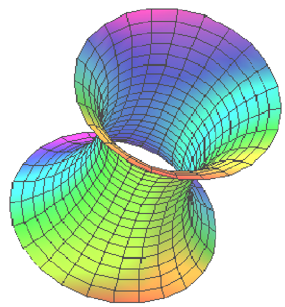


Figure 4.4

This surface of revolution is of a negative Gaussian curvature $K=-a^{-2}$

Symmetries of heat equation on a hyperboloid of one sheet

For $f(x)=\ln b\cosh rx$, $f_{xx}=r^2\,\mathrm{sech}^2 rx$ i.e. $f_{xx}>0\ \forall x$ and $n^2=r^2b^2$ therefore we have the following infinitesimals.

$$\xi = k_1e^{-rby}+k_2e^{rby}$$

$$\tau = k_3$$

$$\vartheta = \frac{1}{b}\tanh(rx)(k_1e^{-rby}-k_2e^{rby})+k_4$$

$$\varphi = k_5u+g(x,y,t).$$

Thus the corresponding symmetry algebra is given by;

$$X_1=e^{-rby}\frac{\partial}{\partial x}+\frac{1}{b}e^{-rby}\tanh(rx)\frac{\partial}{\partial y},\ \ X_3=\frac{\partial}{\partial t},\quad X_5=u\frac{\partial}{\partial u},$$

$$X_2=e^{rby}\frac{\partial}{\partial x}-\frac{1}{b}e^{rby}\tanh(rx)\frac{\partial}{\partial y},\quad X_4=\frac{\partial}{\partial y},\ \ X_g=g\frac{\partial}{\partial u}.$$

Table 4.4:

Commutator table for the symmetry algebra of heat equation on a hyperboloid of one sheet

	X_1	X_2	X_3	X_4	X_5
X_1	0	$2rb^{-1}X_4$	0	$-brX_1$	0
X_2	$-2rb^{-1}X_4$	0	0	brX_2	0
X_3	0	0	0	0	0
X_4	brX_1	$-brX_2$	0	0	0
X_5	0	0	0	0	0

This completes our analysis of the solutions of the equation (4.2) with $k \neq 0$. We observe that the determining functions $f(x)$ under this case are all of constant curvature surfaces of revolution.

Case 2 $k = 0$ this implies by equation (4.2) that

$$f_x^2 + f_{xx} = 0 \tag{4.3}$$

and by e_{17} that is if $\xi \neq 0$

$$f_x f_{xxx} - 2f_{xx}^2 = 0 \tag{4.4}$$

The solution of (4.3) and (4.4) is the form

$$f(x) = \ln|c_1 x + c_2|$$

This determining function gives all the three flat surfaces of revolution. For $c_1 = 0$ and $c_1 = 1$, we respectively have cylinder and a plane, otherwise we have a cone. The symmetry analysis of these surfaces was discussed in chapter 3.

4.1.2. Classification for the case $\xi_y = 0$

This results into two possible cases

Case 1. $\xi = 0$

This corresponds to the minimal symmetry algebra.

Case 2. $\xi \neq 0$

Using e_{17}, we obtain equation (4.4), that is

$$f_x f_{xxx} - 2f_{xx}^2 = 0$$

Since we have already discussed the case where $f_{xx} = 0$ in section 4.1.1 case 1.1, we only a only consider a case where $f_{xx} \neq 0$.

Dividing Eq(4.4) throughout by $f_x f_{xx}$ gives

$$\frac{2f_{xx}}{f_x} - \frac{f_{xxx}}{f_{xx}} = 0$$

Integrating with respect to x gives

$$\ln f_x^2 - \ln |f_{xx}| = k$$

$$|f_{xx}| = e^{-k} f_x^2$$

we can now write the above equation in the form

$$f_{xx} = -a^{-1} f_x^2 \quad \text{for some } a \neq 0$$

By separation of variables

$$1 / f_x = a^{-1}x - c$$

Therefore the solution is given by

$$f(x) = \ln[b(x-c)^a]; \quad b > 0, \ x > c, \ a \neq 0,1$$

Implying that $w(x) = b(x-c)^a$

To completely understand the geometry of the surface of revolution we are working with, we need obtain its Gaussian curvature K.

Recall the formula for Gaussian curvature from section 1.1

$$\begin{aligned} K &= -\frac{w''}{w(x)} \\ &= -\frac{a(a-1)}{(x-c)^2}, \text{ for } x > c \end{aligned}$$

This implies this surface of revolution has a positive variable curvature when $a < 0$ and $a > 1$. For $0 < a < 1$, this surface is of a negative variable curvature.

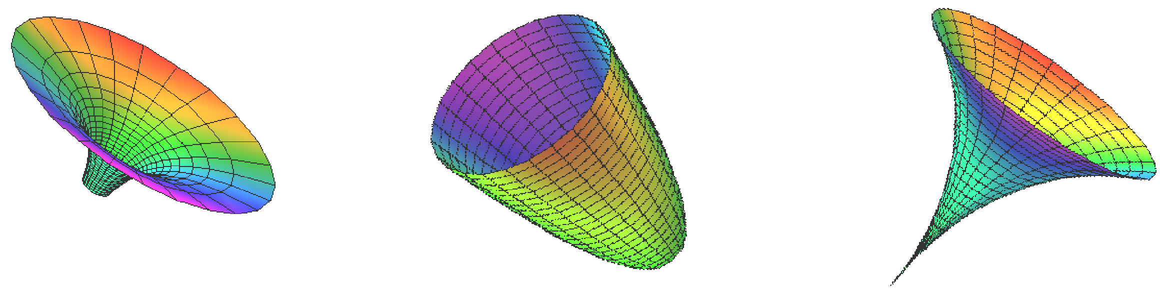
Without loss of generality, let $b = |a^{-1}|$ and $c = 0$ since they don't affect the geometry of this surface of revolution. From the relationship between $v(x)$ and $w(x)$ we note that

$$v(x) = \begin{cases} \int_{\varsigma}^{x} \sqrt{1+t^{a-1}}dt; \; a < 0, & \varsigma < x < \infty, \; \varsigma > 0 \\ \int_{1}^{x} \sqrt{1-t^{a-1}}dt; \; 0 < a < 1, & 1 < x < \infty \\ \int_{0}^{x} \sqrt{1-t^{a-1}}dt; \; a > 1, & 0 < x < 1 \end{cases}$$

Thus the coordinate patch for this surface of revolution takes a form below

$$X(x,y) = \begin{cases} \left(\int_{\varsigma}^{x} \sqrt{1+t^{a-1}}dt, \; -a^{-1}x^{a}\cos y, \; -a^{-1}x^{a}\sin y \right); & a < 0, \; \varsigma < x < \infty, \; \varsigma > 0 \\ \left(\int_{1}^{x} \sqrt{1-t^{a-1}}dt, \; a^{-1}x^{a}\cos y, \; a^{-1}x^{a}\sin y \right); & 0 < a < 1, \; 1 < x < \infty \\ \left(\int_{0}^{x} \sqrt{1-t^{a-1}}dt, \; a^{-1}x^{a}\cos y, \; a^{-1}x^{a}\sin y \right); & a > 0, \; 0 < x < 1 \end{cases}$$

Below are the illustrations of such surfaces of revolution.



$a=-1,\ 0.2<x<2,$ $\qquad$ $a=0.5,\ 1<x<50,$ $\qquad$ $a=2,\ 0<x<0.8$

Figure 4.5

Remarks

Let $S(a)$ *be the surface of revolution corresponding to the curve* $\alpha(x)=(v,w)$

- *If* $a=-1$ *,* $S(a)$ *is Gabriel's Horn.*
- *If* $a=\frac{1}{2}$ *then* $S(a)$ *is an ordinary paraboloid.*
- *If* $a>1$*, then* $S(a)$ *resembles a Pseudosphere but with a variable negative curvature.*

Symmetries of heat equation on this type of surface of revolution

For $f(x)=\ln[b(x-c)^a]$, $(e_6)_x : \xi_{xx}=0$, $(e_6)_y : \xi_{xy}=0$. by e_3 and e_4 respectively

This implies that we have system of determining equations below.

$$e_0 : \xi_u = 0$$
$$e_1 : \vartheta_u = 0$$
$$e_2 : \tau_u = 0$$
$$e_3 : \tau_y = 0$$
$$e_4 : \tau_x = 0$$
$$e_5 : \varphi_{uu} = 0$$
$$e_6 : 2\xi_x - \tau_t = 0$$
$$e_7 : \vartheta_x = 0$$
$$e_8 : 2a(x-c)^{-1}\xi - \tau_t + 2\vartheta_y = 0$$
$$e_9 : b^2x^{2a}[a(x-c)^{-1}\varphi_x + \varphi_{xx} - \varphi_t] + \varphi_{yy} = 0$$

$$e_{10} : b^2(x-c)^{2a}\vartheta_t + 2\varphi_{uy} = 0$$
$$e_{11} : \xi_t + 2\varphi_{xu} = 0$$
$$e_{12} : (x-c)\xi_x - \xi = 0$$
$$e_{13} : \varphi_{ut} + \xi_{xt} = 0$$
$$e_{14} : \xi_{tt} = 0$$

For ξ; $\xi_{tt} = 0$, $(e_6)_x : \xi_{xx} = 0,\ \xi_y = 0$. this implies that $\xi_x = c_1 t + k_1$.

$$e_{12} : (x-c)\xi_x - \xi = 0 \Rightarrow \xi = (c_1 t + k_1)(x-c)$$

For τ; $\tau_u = \tau_y = \tau_x = 0 \Rightarrow \tau = \tau(t)$

$$e_6 : 2\xi_x - \tau_t = 0 \Rightarrow \tau_t = 2(c_1 t + k_1) \Rightarrow \tau = c_1 t^2 + 2k_1 t + k_2$$

For ϑ; $\vartheta_u = 0$ and by e_7, $\vartheta_x = 0 \Rightarrow \vartheta = \vartheta(y,t)$

$$e_8 : (a-1)(c_1 t + k_1) + \vartheta_y = 0 \Rightarrow \vartheta = (1-a)(c_1 t + k_1)y + p(t)$$

For φ; $\varphi_{uu} = 0$ and by e_{13}, $\varphi_{ut} = -c_1 \Rightarrow \varphi_u = -c_1 t + q(x,y)$

$$e_{11} : c_1(x-c) + 2q_x = 0 \ \Rightarrow q(x,y) = -\tfrac{1}{4}c_1(x-c)^2 + h(y)$$
$$e_{10} : c_1 b^2(x-c)^{2a}(1-a)y + p'(t) + 2h'(y) = 0 \Rightarrow p'(t) = c_1 = h'(y) = 0$$

Thus $\varphi_{uu} = \varphi_{ux} = \varphi_{uy} = \varphi_{ut} = 0$ implying that $\varphi = k_4 u + g(x,y,t)$

Therefore the infinitesimals take the form

$$\xi = k_1(x-c)$$
$$\tau = 2k_1 t + k_2$$
$$\vartheta = k_1(1-a)y + k_3$$
$$\varphi = k_4 u + g(x,y,t)$$

with $g(x,y,t)$ satisfying the Eq(1.1)

Thus the corresponding symmetry algebra is given by;

$$X_1 = (x-c)\frac{\partial}{\partial x} + (1-a)y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial t},\ X_3 = \frac{\partial}{\partial y},\ X_4 = u\frac{\partial}{\partial u},\ X_g = g\frac{\partial}{\partial u}.$$

Table 4.5:

Commutator table for the symmetry algebra of heat equation on a surface of revolution with the determining function $f(x) = \ln[b(x-c)^a]$

	X_1	X_2	X_3	X_4
X_1	0	$2X_3$	$(1-a)X_3$	0
X_2	$-2X_3$	0	0	0
X_3	$(a-1)X_3$	0	0	0
X_4	0	0	0	0

This completes our symmetry analysis of the heat equation on surfaces of revolution.

Chapter 5

Classification of non-flat surfaces of revolution according to symmetries of the wave equation

In the previous chapter of our work, we gave a complete classification of non-flat surfaces of revolution according to symmetries of heat equation. In this chapter we apply similar ideas to the wave equation on surfaces of revolution. As already clarified, we give a complete classification of surfaces of revolution according to the symmetries of wave equation.

As in the previous chapter, the analysis consists of first finding Lie symmetries of the heat equation on an arbitrary surface of revolution with a determining function $f(x)$ and then determining all other forms of $f(x)$ for which larger symmetry groups exist. Precisely we obtain the following results.

Theorem 5.1

The minimal symmetry algebra of the wave equation

$$u_{tt} = f'(x)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + e^{-2f(x)}\frac{\partial^2 u}{\partial y^2}$$

on a surface of revolution is generated by

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = u\frac{\partial}{\partial u}$$

and is obtained for an arbitrary determining function f. The larger symmetry algebra exists in the cases given in the table 1.2

5.1. Symmetry analysis of wave equation on non-flat surface of revolution

In chapter one, we saw that the wave equation on any surface of revolution parameterized by the coordinate patch

$$X(x,y) = (v(x),\ \ e^{f(x)}\cos y,\ \ e^{f(x)}\sin y),\ 0 \le y < 2\pi.$$

is of the form

$$u_{tt} = f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy};\ \ w(x) = e^{f(x)}. \tag{1.2}$$

Recall from section 2.2.2, that the symmetry generator of the Lie algebra of the wave equation on any surface of revolution is of the form

$$X = \xi(x,y,t,u)\frac{\partial}{\partial x} + \vartheta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \varphi(x,y,t,u)\frac{\partial}{\partial u}$$

Furthermore we denoted the second prolongation by $X^{[2]}$ and by applying the invariance criterion,

$$X^{[2]}(u_{tt} - f'(x)u_x - u_{xx} - e^{-2f(x)}u_{yy})\Big|_{u_{tt}=f'(x)u_x+u_{xx}+e^{-2f(x)}u_{yy}} = 0$$

we obtained the following system of thirteen determining equations.

$$e_1 : \xi_u = 0$$
$$e_2 : \vartheta_u = 0$$
$$e_3 : \tau_u = 0$$
$$e_4 : \varphi_{uu} = 0$$
$$e_5 : \xi_t - \tau_x = 0$$
$$e_6 : \xi_x - \tau_t = 0$$
$$e_7 : \vartheta_t - e^{-2f}\tau_y = 0$$
$$e_8 : \vartheta_x + e^{-2f}\xi_y = 0$$
$$e_9 : \xi f_x - \xi_x + \vartheta_y = 0$$
$$e_{10} : \varphi_{tt} - f_x\varphi_x - \varphi_{xx} - e^{-2f}\varphi_{yy} = 0$$
$$e_{11} : \tau_x f_x + e^{-2f}\tau_{yy} + 2\varphi_{ut} = 0$$

$$e_{12}: \xi f_{xx} + f_x \xi_x + 2\varphi_{xu} - e^{-2f}\xi_{yy} = 0$$
$$e_{13}: \vartheta_{tt} - \vartheta_{xx} - f_x \vartheta_x + 2e^{-2f}\varphi_{uy} - e^{-2f}\vartheta_{yy} = 0$$

Next we carry out the triangulation process of the determining equations as follows to obtain the minimal symmetry algebra.

Using $(e_6)_x - (e_5)_t$ and $(e_6)_t - (e_5)_x$ we obtain the following

$$e_{14}: \tau_{xx} - \tau_{tt} = 0$$
$$e_{15}: \xi_{xx} - \xi_{tt} = 0$$

Using $(e_7)_t$, e_8, $(e_8)_x$, $(e_8)_x$ that is, substituting the derivatives of ϑ in e_{13} simplifies it to

$$e_{13}: \xi_{xy} + 2\varphi_{uy} = 0$$

Using $(e_7)_x - (e_8)_t$, we obtain

$$e_{16}: f_x \tau_y - \xi_{ty} = 0$$

Using $(e_7)_y - (e_9)_t$ we obtain

$$e_{17}: \xi_{tx} - e^{-2f}\tau_{yy} - \xi_t f_x = 0$$

This implies that

$$e_{11}: \xi_{tx} + 2\varphi_{ut} = 0$$

By $(e_9)_x - (e_8)_y$ we observe that

$$e_{18}: \xi_x f_x + \xi f_{xx} - \xi_{xx} - e^{-2f}\xi_{yy} = 0$$

By $(e_{10})_u$, e_{11}, e_{12}, e_{13}, we note that

$$-\xi_{ttx} + \xi_{xxx} + f_x \xi_{xx} + e^{-2f}\xi_{xyy} = 0$$

and by e_{15} we have

$$e_{19} : f_x \xi_{xx} + e^{-2f} \xi_{xyy} = 0$$

By $(e_6)_{yy}$ and e_{19} we note that

$$e^{-2f} \tau_{tyy} = e^{-2f} \xi_{xyy} = -f_x \xi_{xx}$$

Using above result, $(e_{17})_t$ and e_{17} gives

$$(e_{17})_t \colon \xi_{ttx} = e^{-2f} \tau_{tyy} + \xi_{tt} f_x = -f_x \xi_{xx} + \xi_{tt} f_x = 0.$$

$$e_{20} : \xi_{ttx} = 0 \Rightarrow \varphi_{utt} = 0 \text{ and } \xi_{xxx} = 0$$

By $(e_{18})_x + e_{19}$ and e_{20} we note that

$$e_{21} : 2\xi_{xx} f_x + 2\xi_x f_{xx} + \xi f_{xxx} + 2 f_x e^{-2f} \xi_{yy} = 0$$

Eliminating ξ using e_{18} and e_{21} gives

$$e_{22} : \xi_x (2 f_{xx}^2 - f_x f_{xxx}) + (2 f_x f_{xx} + f_{xxx})(\xi_{xx} + e^{-2f} \xi_{yy}) = 0$$

Eliminating ξ_x using e_{18} and e_{21} gives

$$e_{23} : \xi (f_x f_{xxx} - 2 f_{xx}^2) + 2(\xi_{xx} + e^{-2f} \xi_{yy})(f_x^2 + f_{xx}) = 0$$

Similarly eliminating $(\xi_{xx} + e^{-2f} \xi_{yy})$ using e_{22} and e_{23} gives

$$e_{24} : \xi (2 f_x f_{xx} + f_{xxx}) + 2\xi_x (f_x^2 + f_{xx}) = 0$$

Differentiating e_{24} twice with respect to t and using e_{15} gives;

$$e_{25} : (2 f_x f_{xx} + f_{xxx}) \xi_{xx} = 0.$$

Without any restriction on $f(x)$, we note that $\xi_{xx} = 0$. It then follows immediately from $(e_{16})_t$ that $\xi_{xy} = 0$.

Differentiating e_{24} once with respect to y gives

$$(2 f_x f_{xx} + f_{xxx}) \xi_y = 0$$

This implies that $\xi_y = 0$ and $\xi = 0$ by e_{23}.

By e_3, e_5, e_6, and e_{11} we note that

$$\tau = ly + k_1$$

By e_2, e_8, e_9, and e_{13} we note that

$$\vartheta = rt + k_2.$$

and by e_7, $r - le^{-2f} = 0 \Rightarrow l = r = 0$. Hence $\vartheta = k_2$ and $\tau = k_1$

By e_4, e_{11}, e_{12}, and e_{13}, we note that

$$\varphi = k_3 u + h(x, y, t).$$

with $h(x, y, t)$ satisfying equation (1.2)

Hence the minimal symmetry algebra is three dimensional which exist for any choice of $f(x)$

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = u\frac{\partial}{\partial u} \text{ and } X_h = h\frac{\partial}{\partial u}.$$

Our next task is to investigate the surfaces of revolution on which the wave equation might have a larger algebra. To avoid repeating the results from chapter 3, we consider only the non-flat surfaces of revolution.

From theorem 1.1, the Gaussian curvature K of a surface of revolution generated by a unit speed curve $\alpha(x) = (v(x), e^{f(x)})$ is given by $f_{xx} + f_x^2 + K = 0$ with $K \neq 0$ for non-flat surfaces of revolution.

Using $\xi_{xx} e_{24}$ and e_{25}, we obtain $\xi_x \xi_{xx}(f_x^2 + f_{xx}) = 0$ implying that for non-flat surfaces of revolution $\xi_x \xi_{xx} = 0$ and consequently $\xi_{xx} = 0$. It then follows immediately from $(e_{16})_t$ and e_{15} that $\xi_{xy} = 0$ and $(e_{24})_y$ gives $(2f_x f_{xx} + f_{xxx})\xi_y = 0$.

To look for the determining function $f(x)$ which may give larger symmetry algebra, we consider the cases $\xi_y \neq 0$ and $\xi_y = 0$ in sections 5.1.1 and 5.1.2 respectively.

5.1.1. Classification for the case $\xi_y \neq 0$

By $(e_{24})_y$, we note that

$$2f_x f_{xx} + f_{xxx} = 0 \qquad (5.1)$$

It then follows directly by integration that for some constant $k \neq 0$

$$f_{xx} + f_x^2 = k \qquad (5.2)$$

For this case we shall first give a general analysis of the symmetries of the wave equation on surface of revolution whose determining function $f(x)$ satisfies the equations (5.1) and (5.2) with $k \neq 0$.

By $(e_{16})_x + f_x e_{16}$ we note that $(f_x^2 + f_{xx})\tau_y = k\tau_y = 0$ implying that $\tau_y = 0$.

If $f(x)$ satisfies the equations (5.2) with $k \neq 0$ i.e. $f_x^2 + f_{xx} = k,\ k \neq 0$, then by e_{24} we note that $\xi_x = 0$ and $f_x \neq 0$, implies by e_{17} that $\xi_t = 0$

The system of determining equations in this case therefore takes a form

$$e_1 : \xi_u = 0$$
$$e_2 : \vartheta_u = 0$$
$$e_3 : \tau_u = 0$$
$$e_4 : \varphi_{uu} = 0$$
$$e_5 : \tau_x = 0$$
$$e_7 : \vartheta_t = 0$$
$$e_8 : \vartheta_x + e^{-2f}\xi_y = 0$$
$$e_9 : \xi f_x + \vartheta_y = 0$$
$$e_{10} : \varphi_{tt} - f_x\varphi_x - \varphi_{xx} - e^{-2f}\varphi_{yy} = 0$$

$$e_{11} : \varphi_{ut} = 0$$
$$e_{12} : \varphi_{xu} = 0$$
$$e_{13} : \varphi_{uy} = 0$$
$$e_{18} : \xi f_{xx} - e^{-2f}\xi_{yy} = 0$$

For ξ, we note that $\forall f(x);\ 2f_x f_{xx} + f_{xxx} = 0,\ e^{2f} f_{xx}$ is a constant function. It then follows that $\xi = \xi(y)$. By $e^{2f}e_{18}$ and for $n \in \mathbb{R}$, we have the following.

For $f_{xx} < 0 \Rightarrow e^{2f} f_{xx} = -n^2 \Rightarrow \xi(y) = k_1 \sin(ny) + k_2 \cos(ny)$
For $f_{xx} = 0 \Rightarrow e^{2f} f_{xx} = 0 \Rightarrow \xi(y) = k_1 y + k_2$
For $f_{xx} > 0 \Rightarrow e^{2f} f_{xx} = n^2 \Rightarrow \xi(y) = k_1 e^{ny} + k_2 e^{-ny}$

For $\tau;$ we observe that $\tau = k_3$

For $\vartheta;\ e_7 : \vartheta_t = 0,\ e_2 : \vartheta_u = 0,\ e_8 : \vartheta_x + e^{-2f}\xi_y = 0$ and $e_9 : \xi f_x + \vartheta_y = 0$
$\vartheta_t = \vartheta_u = 0$ implies that $\vartheta = \vartheta(x, y)$
By e_8 we note that

$$\vartheta(x, y) + \xi_y \int e^{-2f(x)} dx = p(y)$$

Using e_9, we obtain

$$p_y = \xi_{yy} \int e^{-2f(x)} dx - \xi f_x$$

implying that

$$p(y) = \int \left(\xi_{yy} \int e^{-2f(x)} dx - \xi f_x \right) dy + k_4$$

Therefore

$$\vartheta(x, y) = \int \left(\xi_{yy} \int e^{-2f(x)} dx - \xi f_x \right) dy - \xi_y \int e^{-2f(x)} dx + k_4$$

For $\varphi;\ \varphi_{uu} = 0,\ \varphi_{ux} = 0,\ \varphi_{uy} = 0,\ \varphi_{ut} = 0$ this implies that $\varphi = k_5 u + h(x, y, t)$ with $h(x, y, t)$ satisfying equation (1.2)

Hence the symmetry algebra of the wave in this case is five dimensional similar to that of the heat equation.

In our next discussion, we look at the different solutions of the equation 5.2 with $k \neq 0$.

Regarding the commutator tables in each case we shall refer to a case giving similar results in chapter 4.

Case 1. $f(x)$ is linear i.e. $f_{xx} = 0$ and $f_x \neq 0$

This implies that $f(x) = m^{-1}x + c$ for some $m \neq 0$ and c.

If without loss of generality, we let $c = \ln|m|$, then $w(x) = |m|e^{x/m}$

It then follows from the relationship between $v(x)$ and $w(x)$ that

$$v(x) = \int_0^x \sqrt{1 - e^{2t/m}}\,dt; \quad -\infty < x < 0 \; if \;\; m > 0 \text{ and } 0 \leq x < \infty \;\; if \;\; m < 0.$$

If we further let $|m| = a$ for some $a > 0$, then curve $\alpha(x)$ takes the form

$$\alpha(x) = \begin{cases} \left(\int_0^x \sqrt{1 - e^{-2t/a}}\,dt, \;\; a\,e^{-x/a}\right); & 0 \leq x < \infty \\ \left(\int_0^x \sqrt{1 - e^{2t/a}}\,dt, \quad a\,e^{x/a}\right); & -\infty < x < 0 \end{cases}$$

This is a unit speed parameterization of the ***tractrix*** and its corresponding surface of revolution is a ***Pseudosphere*** or ***tractoid***. [15]

The coordinate patch for a ***Pseudosphere*** or ***tractoid*** is of the form

$$X(x,y) = \begin{cases} \left(\int_0^x \sqrt{1 - e^{-2t/a}}\,dt, \;\; ae^{-x/a}\cos y, \;\; ae^{-x/a}\sin y\right); & 0 \leq y < 2\pi \\ \left(\int_0^x \sqrt{1 - e^{2t/a}}\,dt, \;\; ae^{x/a}\cos y, \;\; ae^{x/a}\sin y\right); & 0 \leq y < 2\pi \end{cases}$$

Below is an illustration of an example of a Pseudosphere or tractoid with $a = 4$ and $-8 \leq x \leq 8$.

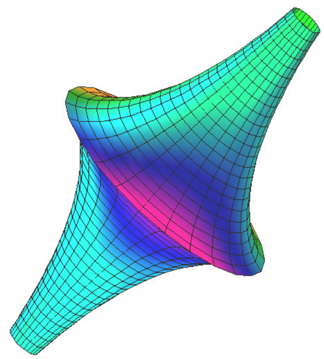


Figure 5.1

The Gaussian curvature for the tractoid is negative and given by $K = -m^{-2}$

Symmetries of the wave equation on a Pseudosphere or tractoid

We now let $m^{-1} = b$. Thus $f(x) = bx + c$ and $w(x) = e^{bx+c}$ implying that

$$\xi = k_1 y + k_2.$$
$$\tau = k_3.$$
$$\vartheta = \frac{k_1}{2b} e^{-2(bx+c)} - \frac{1}{2} by(k_1 y + 2k_2) + k_4$$
$$\varphi = k_5 u + h(x, y, t).$$

Hence the symmetry algebra is spanned by

$$X_1 = y\frac{\partial}{\partial x} + \left(\tfrac{1}{2b} e^{-2(bx+c)} - \tfrac{1}{2} by^2\right)\frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial x} - by\frac{\partial}{\partial y},\ X_3 = \frac{\partial}{\partial t},\ X_4 = u\frac{\partial}{\partial u}$$
$$X_5 = \frac{\partial}{\partial y},\ X_h = h\frac{\partial}{\partial u}$$

Case 2. $f(x)$ is non-linear i.e. $f_{xx} \neq 0$.

The substitution $f_x = H(x)$ reduces equation (5.2) to

$$H^2 + H_x = k.$$

Next we analyze different cases resulting from different values of k.

Case 2.1, $k < 0$, we let $k = -m^2$ for some number $m > 0$

$$H^2 + H_x = -m^2$$

$$H = f_x = -m\tan(mx + mc)$$

This gives

$$f(x) = \ln b|\cos(mx + mc)|; \quad b > 0$$

Let $m = a^{-1}$ and without loss of generality, we consider

- $a > 0,\ c = 0,$ and thus $w(x) = b|\cos(a^{-1}x)|$.
- One case where $w(x) > 0,\ w(x) = b\cos(a^{-1}x)$ and the parameter x has the following ranges.

$$\begin{aligned}
&\mathit{if}\ \ b = a,\ \ \mathit{then}\ \ -\tfrac{1}{2}\pi a < x < \tfrac{1}{2}\pi a;\\
&\mathit{if}\ \ b < a,\ \ \mathit{then}\ \ -\tfrac{1}{2}\pi a < x < \tfrac{1}{2}\pi a;\\
&\mathit{if}\ \ b > a,\ \ \mathit{then}\ \ -a\arcsin\frac{a}{b} < x < a\arcsin\frac{a}{b}
\end{aligned}$$

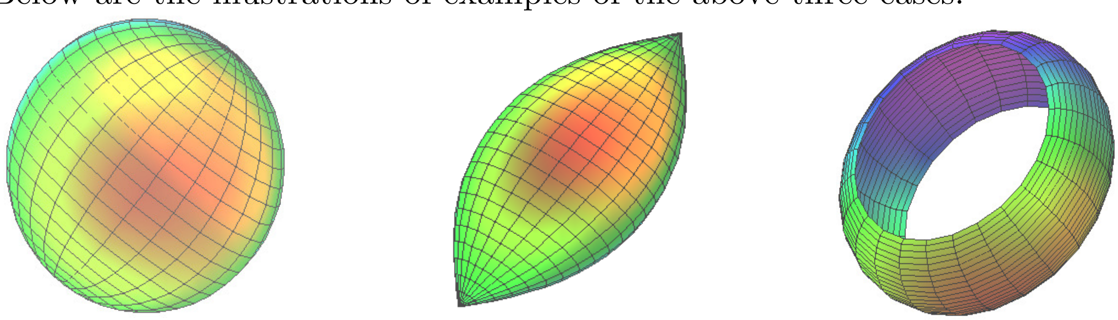
It then follows from the relationship between $v(x)$ and $w(x)$ that

$$v(x) = \int_0^{x/a}\sqrt{a^2 - b^2\sin^2 t}\,dt.$$

Therefore the coordinate patch of the above surface is given by

$$X(x,y) = \left(\int_0^{x/a}\sqrt{a^2 - b^2\sin^2 t}\,dt,\ \ b\cos(a^{-1}x)\cos y,\ \ b\cos(a^{-1}x)\sin y\right)$$

Below are the illustrations of examples of the above three cases.



when $a = b = 1$ *when* $a = 1.5,\ b = 1$ *when* $a = 1,\ b = 1.5.$

Figure 5.2

Remarks

Let $S(a,b)$ *be the surface of revolution corresponding to the curve* $\alpha(x)=(v,w)$

- $S(a,a)$ *is an ordinary sphere of radius a*
- *(Spindle type) if* $0<b<a$ *then* $S(a,b)$ *is a surface of revolution is like a rugby ball with sharp vertices on its axis of revolution.*
- *(Bulge type) if* $0<a<b,$ *then* $S(a,b)$ *is a barrel - shaped and does not meet axis of revolution.* [15]

It can be easily observed that this surface of revolution has a positive Gaussian curvature i.e. $K=a^{-2}$

Symmetries of the wave equation on a the surface $S(a,b)$

For $f(x)=\ln b|\cos(mx)|$, $f_{xx}=-m^2\sec^2 mx$ i.e. $f_{xx}<0\ \forall x$ and $n^2=m^2b^2$ therefore

we have the following infinitesimals.

$$\xi=k_1\sin(mby)+k_2\cos(mby)$$
$$\tau=k_3$$
$$\vartheta=k_4-\frac{1}{b}\tan(mx)(k_1\sin(mby)-k_2\cos(mby))$$
$$\varphi=k_5u+h(x,y,t).$$

Consequently the corresponding symmetry algebra is given by

$$X_1=\sin(bmy)\frac{\partial}{\partial x}-\frac{1}{b}\tan mx\cos(bmy)\frac{\partial}{\partial y},\ \ X_3=\frac{\partial}{\partial t},\ \ X_5=u\frac{\partial}{\partial u}$$
$$X_2=\cos(bmy)\frac{\partial}{\partial x}+\frac{1}{b}\tan mx\sin(bmy)\frac{\partial}{\partial y},\ \ X_4=\frac{\partial}{\partial y},\ \ X_h=h\frac{\partial}{\partial u}$$

Case 2.2 When $k>0$ we let $k=r^2$

$$H^2+H_x=r^2$$

$$\ln\left|\frac{H+r}{H-r}\right| = 2r(x+p)$$

This gives two possible cases below.

Case 2.2.1, when $H > |r|$

$$\ln\left(\frac{H+r}{H-r}\right) = 2r(x+p)$$

$$H = f_x = \frac{r(1+e^{2r(x+p)})}{e^{2r(x+p)}-1}$$

This implies that

$$f(x) = \ln b\,|\sinh(r(x+p))|; \quad b > 0$$

Let $r = a^{-1}$, $a \neq 0$, and without loss of generality

- We consider $a > 0$, $p = 0$, and thus $w(x) = b|\sinh(a^{-1}x)|$ implying that
- We further consider the first case $w(x) = b\sinh(a^{-1}x)$; $x > 0$.

It then follows from the relationship between $v(x)$ and $w(x)$ that

$$v(x) = \int_0^{x/a}\sqrt{a^2 - b^2\cosh^2 t}\,dt; \; b \leq a \text{ and } 0 < x \leq a\,\mathrm{arcsinh}\left(\frac{\sqrt{b^2-a^2}}{b}\right)$$

Therefore this surface of revolution has a parameterization.

$$X(x,y) = \left(\int_0^{x/a}\sqrt{a^2 - b^2\cosh^2 t}\,dt,\; b\sinh(a^{-1}x)\cos y,\; b\sinh(a^{-1}x)\sin y\right)$$

Below is an example of an illustration such a surface of revolution and it's of a ***conic type*** [15] *With* $a = 4$ *and* $b = 2$.

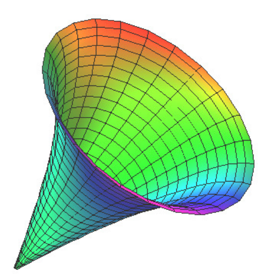


Figure 5.3

The Gaussian curvature is given by $K = -a^{-2}$, i.e. negative and constant.

Symmetries of wave equation on a conic type surface

For $f(x) = \ln b|\sinh(rx)|$, $f_{xx} = -r^2 \operatorname{sech}^2 rx$ i.e. $f_{xx} < 0\ \forall x$ and $n^2 = r^2 b^2$ therefore we have the following infinitesimals.

$$\xi = k_1 \sin(rby) + k_2 \cos(rby)$$
$$\tau = k_3$$
$$\vartheta = k_4 + \frac{(k_1 \sin(rby) - k_2 \cos(rby))}{b \tanh rx}$$
$$\varphi = k_5 u + h(x, y, t).$$

Thus the corresponding symmetry algebra is given by;

$$X_1 = \sin(bry)\frac{\partial}{\partial x} + \frac{\cos(bry)}{b\tanh(rx)}\frac{\partial}{\partial y},\ X_3 = \frac{\partial}{\partial t},\quad X_5 = u\frac{\partial}{\partial u},$$
$$X_2 = \cos(bry)\frac{\partial}{\partial x} - \frac{\sin(bry)}{b\tanh(rx)}\frac{\partial}{\partial y},\ X_4 = \frac{\partial}{\partial y},\ X_h = h\frac{\partial}{\partial u}.$$

Case 2.2.2, when $H < |r|$

$$\ln\left(\frac{H + r}{r - H}\right) = 2r(x + p)$$
$$H = f_x = \frac{r(e^{2r(x+p)} - 1)}{e^{2r(x+p)} + 1}$$

This implies that

$$f(x) = \ln b \cosh(r(x + p)); \quad b > 0$$

Let $r = a^{-1}$, $a \neq 0$, and without loss of generality we consider $p = 0$

It then follows from the relationship between $v(x)$ and $w(x)$ that

$$v(x) = \int_0^{x/a} \sqrt{a^2 - b^2 \sinh^2 t}\,dt; \quad -a \operatorname{arcsinh}\frac{b}{a} < x \leq a \operatorname{arcsinh}\frac{b}{a}$$

Therefore this surface of revolution has parameterization

$$X(x,y) = \left(\int_0^{x/a} \sqrt{a^2 - b^2 \sinh^2 t}\,dt,\ \ b\cosh(a^{-1}x)\cos y,\ \ b\sinh(a^{-1}x)\sin y \right)$$

Below is an example of an illustration such a surface of revolution and it's of a ***Hyperboloid of one sheet type*** [15]. *With* $a = 4$ and $b = 2$

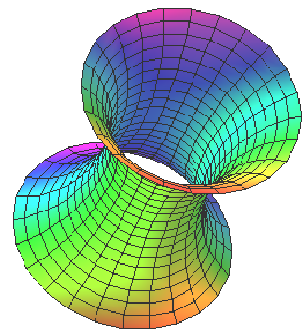


Figure 5.4

The Gaussian curvature is given by $K = -a^{-2}$, i.e. negative Gaussian curvature.

Symmetries of wave equation on a hyperboloid of one sheet

For $f(x) = \ln b\cosh rx$, $f_{xx} = r^2\,\mathrm{sech}^2 rx$ i.e. $f_{xx} > 0\ \forall x$ and $n^2 = r^2 b^2$ therefore we have the following infinitesimals.

$$\xi = k_1 e^{-rby} + k_2 e^{rby}$$

$$\tau = k_3$$

$$\vartheta = \frac{1}{b}\tanh(rx)(k_1 e^{-rby} - k_2 e^{rby}) + k_4$$

$$\varphi = k_5 u + h(x,y,t).$$

Thus the corresponding symmetry algebra is given by;

$$X_1 = e^{-rby}\frac{\partial}{\partial x} + \frac{1}{b}e^{-rby}\tanh(rx)\frac{\partial}{\partial y},\ \ X_3 = \frac{\partial}{\partial t},\quad X_5 = u\frac{\partial}{\partial u},$$

$$X_2 = e^{rby}\frac{\partial}{\partial x} - \frac{1}{b}e^{rby}\tanh(rx)\frac{\partial}{\partial y},\quad X_4 = \frac{\partial}{\partial y},\ \ X_h = h\frac{\partial}{\partial u}.$$

This completes our analysis of the solutions of the equation (5.2) with $k \neq 0$. We observe that the determining functions $f(x)$ that satisfy the equation 5.2 with $k \neq 0$ are all of constant curvature surfaces of revolution.

5.1.2. Classification for the case $\xi_y = 0$

This results into two possible cases

Case 1. $\xi = 0$

This corresponds to the minimal symmetry algebra.

Case 2. $\xi \neq 0$

By e_{23} we note that

$$2f_{xx}^2 - f_x f_{xxx} = 0 \tag{5.3}$$

Since we already discussed the case where $f_{xx} = 0$ in section 5.1.1 case 1.1, we only a only consider a case where $f_{xx} \neq 0$.

Dividing Eq(5.3) throughout by $f_x f_{xx}$ gives

$$\frac{2f_{xx}}{f_x} - \frac{f_{xxx}}{f_{xx}} = 0$$

Integrating with respect to x gives

$$\ln f_x^2 - \ln |f_{xx}| = k$$

$$|f_{xx}| = e^{-k} f_x^2$$

we can now write the above equation in the form

$$f_{xx} = -a^{-1} f_x^2 \quad \text{for some } a \neq 0$$

By separation of variables

$$\frac{1}{f_x} = a^{-1}x - c$$

Therefore the solution is given by.

$$f(x) = \ln[b(x-c)^a]; \quad b > 0, \; x > c, \; a \neq 0,1$$

Implying that $w(x) = b(x-c)^a$

To completely understand the geometry of the surface of revolution we are working with, we need obtain its Gaussian curvature K.
Recall the formula for Gaussian curvature from section 1.1

$$\begin{aligned} K &= -\frac{w''}{w(x)} \\ &= -\frac{a(a-1)}{(x-c)^2}, \text{ for } x > c \end{aligned}$$

This implies this surface of revolution has a positive variable curvature when $a < 0$ and $a > 1$. For values of $0 < a < 1$, this surface is of a negative variable curvature.

Without loss of generality, let $b = |a^{-1}|$ and $c = 0$ since these don't affect the geometry of this surface of revolution.

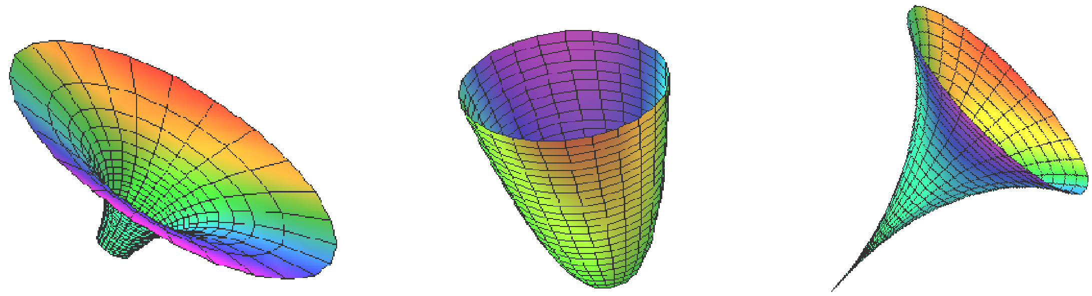
It then follows from the relationship between $v(x)$ and $w(x)$ that

$$v(x) = \begin{cases} \int_{\varsigma}^{x} \sqrt{1+t^{a-1}}dt; \; a < 0, \quad \varsigma < x < \infty, \; \varsigma > 0 \\ \int_{1}^{x} \sqrt{1-t^{a-1}}dt; \; 0 < a < 1, \qquad 1 < x < \infty \\ \int_{0}^{x} \sqrt{1-t^{a-1}}dt; \; a > 1, \qquad\quad 0 < x < 1 \end{cases}$$

Thus the coordinate patch for this surface of revolution takes a form

$$X(x,y) = \begin{cases} \left(\int_{\varsigma}^{x} \sqrt{1+t^{a-1}}dt, \; -a^{-1}x^a \cos y, \; -a^{-1}x^a \sin y \right); \; a < 0, \; \varsigma < x < \infty, \; \varsigma > 0 \\ \left(\int_{1}^{x} \sqrt{1-t^{a-1}}dt, \quad a^{-1}x^a \cos y, \quad a^{-1}x^a \sin y \right); \qquad 0 < a < 1, \; 0 < x < 1 \\ \left(\int_{0}^{x} \sqrt{1-t^{a-1}}dt, \quad a^{-1}x^a \cos y, \quad a^{-1}x^a \sin y \right); \qquad a > 0, \quad 0 < x < 1 \end{cases}$$

Below are the illustrations of such surfaces of revolution.



$a = -1,\ 0.2 < x < 2,$ $\qquad$ $a = 0.5,\ 1 < x < 50,$ $\qquad$ $a = 2,\ 0 < x < 0.8$

Figure 5.5

Remarks

Let $S(a)$ be the surface of revolution corresponding to the curve $\alpha(x) = (v, w)$

- *If $a = -1$, $S(a)$ is Gabriel's Horn.*
- *If $0 < a < 1$ then $S(a)$ is an ordinary paraboloid.*
- *If $a > 1$, then $S(a)$ resembles a Pseudosphere but with a long tail and variable negative curvature.*

Symmetries of wave equation on this type of surface of revolution

For $f(x) = \ln[b(x-c)^a]$, we have the following system of determining equation.

$$e_1 : \xi_u = 0$$
$$e_2 : \vartheta_u = 0$$
$$e_3 : \tau_u = 0$$
$$e_4 : \varphi_{uu} = 0$$
$$e_5 : \xi_t - \tau_x = 0$$
$$e_6 : \xi_x - \tau_t = 0$$
$$e_7 : \vartheta_t = 0$$
$$e_8 : \vartheta_x = 0$$

$$e_9 : a\xi - [\xi_x + \vartheta_y](x-c) = 0$$

$$e_{10} : \varphi_{tt} - \frac{a}{(x-c)}\varphi_x - \varphi_{xx} - \frac{1}{b^2(x-c)^a}\varphi_{yy} = 0$$

$$e_{11} : \xi_{tx} + 2\varphi_{ut} = 0$$

$$e_{12} : \varphi_{xu} = 0$$

$$e_{13} : \varphi_{uy} = 0$$

$$e_{14} : \tau_{xx} - \tau_{tt} = 0$$

$$e_{15} : \xi_{tt} = 0$$

$$e_{17} : (x-c)\xi_{tx} - a\xi_t = 0$$

$$e_{16} : \tau_y$$

$$e_{18} : (x-c)\xi_x - \xi = 0$$

Using $e_{17} - (e_{18})_t$ we note that $\xi_t = 0$

For ξ, $\xi_{xx} = \xi_t = \xi_y = \xi_u = 0$ and $\xi - (x-c)\xi_x = 0$ this implies that

$$\xi = k_1(x-c).$$

For τ, $\tau_x = \tau_y = \tau_u = 0$ and $\xi_x - \tau_t = 0$ imply that

$$\tau = k_1 t + k_2$$

For ϑ; $\vartheta_t = 0, \vartheta_x = 0, (a-1)k_1 + \vartheta_y = 0$ by $e_7\ e_8\ e_9$ respectively, implying that

$$\vartheta = k_1(1-a)y + k_3$$

For φ; $\varphi_{uu} = \varphi_{ut} = 0 = \varphi_{xu} = \varphi_{uy} = 0$, this implies that

$$\varphi = k_4 u + g(x,y,t)$$

Thus the corresponding symmetry algebra is given by;

$$X_1 = x\frac{\partial}{\partial x} - (a-1)y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial t},\ X_3 = \frac{\partial}{\partial y},\ X_4 = u\frac{\partial}{\partial u}\ \ X_g = g\frac{\partial}{\partial u}$$

Table 5.1:

Commutator table for the symmetry algebra of wave equation on a surface of revolution with the determining function $f(x) = \ln[b(x-c)^a]$

	X_1	X_2	X_3	X_4
X_1	0	X_3	$(1-a)X_3$	0
X_2	$-X_3$	0	0	0
X_3	$(a-1)X_3$	0	0	0
X_4	0	0	0	0

With this, we complete our symmetry analysis of the wave equation on surfaces of revolution. In the next chapter, we discuss how to use symmetries to reduce the order of the PDE and consequently give some exact solutions.

Chapter 6

Symmetry reductions and exact solutions

In chapters 4 and 5 we discussed the symmetry algebras for the heat and wave equations on a general surface of revolution. In this chapter, we complete our work by carrying out some symmetry reductions and then determine some analytical solutions of the heat and wave equation on the surface of revolution obtained by solving the reduced PDEs. The symmetry reductions will be performed using the standard method of the introduction of similarity variables. The reduced equations are then tried for solution through some ansatz or other techniques.

6.1. Symmetry reductions and exact solutions of the heat equation on surfaces of revolution.

Recall from chapter 4 that the minimal symmetry algebra for heat equation

$$u_t = f'(x)u_x + u_{xx} + e^{-2f(x)}u_{yy} \tag{1.1}$$

on a general surface of revolution is spanned by the vector fields

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = u\frac{\partial}{\partial u}$$

Table 6.1

Commutator table for minimal Lie symmetry algebra

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	0
X_3	0	0	0

6.1.1. Reduction by 1-dimensional subalgebra.

For 1–dimensional subalgebra, we discuss only two cases below. The first reduction results into an invariant solution of the heat equation with an arbitrary determining function f. For the second reduction, we give solutions for some know surfaces of revolution.

I. Subalgebra $\mathcal{L} = \langle X_1 + aX_2 \rangle,\ a \neq 0$

$$X = \frac{\partial}{\partial y} + a\frac{\partial}{\partial t}$$

The characteristic system of

$$X\mathcal{I} = 0$$

is given by

$$\frac{dt}{a} = \frac{dx}{0} = \frac{dy}{1} = \frac{du}{0}$$

Solving the characteristic system gives the similarity variables below.

$$\xi_1 = x, \quad \xi_2 = y - \frac{t}{a} \ \text{ and } \ V(\xi_1, \xi_2) = u\,.$$

Substitution of the similarity variables in Eq(1.1) and using the chain rule implies the solution of Eq(1.1) is of the form $u = V(\xi_1, \xi_2)$.

We observe that

$$u_x = V_{\xi_1},\ u_{xx} = V_{\xi_1\xi_1},\ u_y = V_{\xi_2},\ u_{yy} = V_{\xi_2\xi_2} \ \text{ and } \ u_t = -\frac{1}{a}V_{\xi_2}\,.$$

Clearly $V(\xi_1, \xi_2)$ satisfies the reduced PDE in 2 independent variables.

$$-\frac{1}{a}V_{\xi_2} = f'(\xi_1)V_{\xi_1} + V_{\xi_1\xi_1} + e^{-2f(\xi_1)}V_{\xi_2\xi_2} \tag{6.2}$$

If we let $V(\xi_1, \xi_2) = c\xi_2 + w(\xi_1)$, then $w(\xi_1)$ satisfies the reduced ODE below.

$$w''(\xi_1) + f'(\xi_1)w'(\xi_1) = k \quad \text{with } k = -(c / a)$$

This implies that

$$w(\xi_1) = k\int \left(e^{-f(\xi_1)}\int e^{f(\xi_1)}d\xi_1 + k_1\right)d\xi_1 + k_2$$

Thus by back substitution we obtain the invariant solution.

$$u(x, y, t) = cy - kt + \int \left(k\int e^{f(x)}dx + k_1\right)e^{f(x)}dx + k_2$$

Solutions of the heat equation on some known surfaces of revolution

- *Cylinder of a unit radius* $(f(x) = 0)$

  $$u(x, y, t) = cy - kt + \tfrac{1}{2}kx^2 + k_1x + k_2$$

- *Unit sphere* $(f(x) = \ln\cos x; -0.5\pi < x < 0.5\pi$

  $$u(x, y, t) = cy - kt - k\ln\cos x + k_1\ln(\sec x + \tan x) + k_2$$

- *Surface of a conic type with* $f(x) = \ln\sinh x;\; x > 0$

  $$u(x, y, t) = cy - kt + k\ln\sinh x - 2k_1\tanh^{-1}e^x + k_2$$

- *Hyperboloid of one sheet* $f(x) = \ln\cosh x$

  $$u(x, y, t) = cy - kt + k\ln\cosh x + 2k_1\tan^{-1}e^x + k_2$$

- *Pseudosphere* or *tractoid* $(f(x) = x)$.

  $$u(x, y, t) = cy - kt + kx - k_1e^{-x} + k_2$$

- *Paraboloid* $f(x) = \tfrac{1}{2}\ln x,\; x > 0$.

  $$u(x, y, t) = cy - kt + 2k_1\sqrt{x} + \frac{1}{3}kx^2 + k_2$$

- $Cone\ (f(x)=\ln lx;\ \ x>0,\ \ l>0,\ \ l\neq 0)$

$$u(x,y,t)=cy-kt+\tfrac{1}{4}kx^2+\frac{k_1\ln x}{l}+k_2$$

- $Torus\ (f(x)=\ln(1+\cos x);\ \ x\in[0,\pi)\cup(\pi,2\pi)$

$$u(x,y,t)=cy-kt+k_1\tan(x/2)+kx\tan(x/2)+k_2$$

II. Subalgebra $\mathcal{L}=\langle aX_2+bX_3\rangle$, $a\neq 0,\ \ b\neq 0.$

$$X=a\frac{\partial}{\partial t}+bu\frac{\partial}{\partial u}$$

The characteristic system of

$$X\mathcal{I}=0$$

is given by

$$\frac{dt}{a}=\frac{dx}{0}=\frac{dy}{0}=\frac{du}{bu}$$

Solving the characteristic system gives the similarity variables below.

$$\xi_1=x,\quad \xi_2=y\ \ \text{and}\ \ V(\xi_1,\xi_2)=\ln u-\frac{b}{a}t\,.$$

Substitution of the similarity variables in Eq(1.1) and using the chain rule implies the solution of Eq(1.1) is of the form $u=e^{\frac{bt}{a}}e^{V(\xi_1,\xi_2)}$

We note that

$$u_x=e^{\frac{bt}{a}}e^{V}V_{\xi_1},\ \ u_{xx}=e^{\frac{bt}{a}}e^{V}(V_{\xi_1}^2+V_{\xi_1\xi_1}),\ \ u_{yy}=e^{\frac{bt}{a}}e^{V}e^{-2f(\xi_1)}(V_{\xi_2}^2+V_{\xi_2\xi_2}),\,u_t=\frac{b}{a}e^{\frac{bt}{a}}e^{V}.$$

Clearly $V(\xi_1,\xi_2)$ is satisfies the PDE in two independent variables.

$$\frac{b}{a}=f'(\xi_1)V_{\xi_1}+V_{\xi_1}^2+V_{\xi_1\xi_1}+e^{-2f(x)}(V_{\xi_2}^2+V_{\xi_2\xi_2}) \tag{6.3}$$

If we let $V(\xi_1,\xi_2)=w(\xi_1)+z(\xi_2)$ then $w(\xi_1)$ and $z(\xi_2)$ respectively satisfy the equations ODEs

$$f'(\xi_1)w'(\xi_1)+w'^2(\xi_1)+w''(\xi_1)=\frac{b}{a}-ce^{-2f(\xi_1)} \tag{6.4}$$

$$z'^2(\xi_2)+z''(\xi_2)=c \tag{6.5}$$

By back substitution of similarity variables, we note that the invariant solution in this case will take the form

$$u(x,y,t) = \exp\left\{\tfrac{b}{a}t + w(x) + z(y)\right\}$$

such that $w(x)$ and $z(y)$ respectively satisfy the equations ODEs below.

$$f'(x)w' + w'^2 + w'' = \frac{b}{a} - ce^{-2f(x)} \tag{6.6}$$

$$z'^2 + z'' = c \tag{6.7}$$

The solution of the Eq(6.7) takes the form

$$z(y) = \begin{cases} \ln k_2 \cosh(ky + k_1); & c > 0, \quad c = k^2 \\ \ln k_2 (y + k_1); & c = 0 \\ \ln k_2 \cos(ky + k_1); & c < 0, \quad c = -k^2 \end{cases}$$

The substitution of $w'(x) = v(x)$ reduces the Eq(6.6) to Riccati ODE

$$f'(x)v + v^2 + v' = \frac{b}{a} - ce^{-2f(x)}$$

whose solution cannot be easily obtained without a particular solution.

In our next discussion, we consider the heat equation on some of the common surfaces of revolution.

- *Cylinder of a unit radius* $(f(x) = 0)$

$$v^2 + v' = \frac{b}{a} - c$$

Depending on the sign of $b/a - c$, we have the following solutions

$$v(x) = \begin{cases} -k_3 \tan(xk_3 + k_4); & b/a - c < 0, \quad b/a - c = -k_3^2 \\ (x + k_4)^{-1}; & b/a - c = 0 \\ k_3 \coth(xk_3 + k_4); & b/a - c > 0, \quad b/a - c = k_3^2 \end{cases}$$

This implies that

$$w(x) = \begin{cases} \ln k_5 \cos(xk_3 + k_4); & b/a - c < 0, \quad b/a - c = -k_3^2 \\ \ln k_5 (x + k_2); & b/a - c = 0 \\ \ln k_5 \sinh(xk_3 + k_4); & b/a - c > 0, \quad b/a - c = k_3^2 \end{cases}$$

Thus the solution takes a form

$$u(x,y,t) = \exp\left\{\tfrac{b}{a} t + w(x) + z(y)\right\}$$

where $w(x)$ and $z(y)$ depends on values of $a,\ b$ and c

- *Cone* $(f(x) = \ln lx;\ x > 0,\ l > 0)$

The Eq(6.6) takes a form

$$x^2 w'' + x^2 w'^2 + x w' = \frac{b}{a} x^2 - \frac{c}{l^2}$$

Letting $w(x) = \ln(\phi(x))$ gives a Bessel's differential equation if $b / a < 0$ and $c \le 0$

$$x^2 \phi''(x) + x\phi'(x) + \phi(x)((mx)^2 - n^2) = 0,\ \ b / a = -m^2\ \ and\ \ c / l^2 = -n^2$$

whose solution takes a form

$$\phi(x) = c_1 J_n(mx) + c_2 Y_n(mx)$$

$$w(x) = \ln\left\{c_1 J_n(mx) + c_2 Y_n(mx)\right\}$$

J_n and Y_n are first and second kind Bessel function of order n.

$$u(x,y,t) = \begin{cases} e^{-m^2 t}(k_3 J_n(mx) + k_4 Y_n(mx))\cos(ky + k_1); c < 0,\ \ c / l^2 = -n^2 \\ e^{-m^2 t}(k_3 J_0(mx) + k_4 Y_0(mx))(ky + k_1); \qquad c = 0 \end{cases}$$

- *Unit sphere* $(f(x) = \ln\cos x; -0.5\pi < x < 0.5\pi$

The Eq(6.6) takes a form

$$w'' + w'^2 - w' \tan x + c \sec^2 x - b / a = 0$$

Putting $r = \sin(x),\ w(r) = \ln(\phi(r))$ reduces the above equation to the form

$$(1 - r^2)\phi''(r) - 2r\phi'(r) + \left(\frac{c}{1 - r^2} - b / a\right)\phi(x) = 0$$

For $c \le 0,\ \ c = -k^2$, we have an associated Legendre ODE with order k and its solution is of the form

$$\phi(r) = c_1 P_m^k(r) + c_2 Q_m^k(r) \quad m(\text{degree}) = \tfrac{1}{2}\left\{(1 - 4b / a)^{1/2} - 1\right\},\ \ a > 4b.$$

This implies that

$$w(x) = \ln(c_1 P_m^k(\sin x) + c_2 Q_m^k(\sin x))$$

Thus the solution takes a form

$$u(x,y,t)=\begin{cases} e^{-m(m+1)t}(k_2P_m^k(\sin x)+k_3Q_m^k(\sin x))\cos(ky+k_1); & c=-k^2 \\ e^{-m(m+1)t}(k_2P_m(\sin x)+k_3Q_m(\sin x))(y+k_1); & c=0 \end{cases}$$

• *Surface of a conic type with* $f(x)=\ln\sinh(x);\ x>0$

The Eq(6.6) takes a form

$$w''+w'^2+w'\coth x+c\operatorname{csch}^2 x-b/a=0$$

Putting $w(x)=\ln(\phi(x)(\sinh x)^{-1/2})$ reduces the above equation to the form

$$\phi''(x)+\left\{\tfrac{1}{4}-\tfrac{1}{2}\coth^2 x+c\operatorname{csch}^2 x-b/a\right\}\phi(x)=0$$

Let $r=\coth x$, this implies that

$$(1-r^2)\phi''(r)-2r\phi'(r)+\left(-c-\frac{1}{4}-\frac{a+4b}{4a(1-r^2)}\right)\phi(r)=0$$

For $c\le 0,\ c=-k^2,$ we have an associated Legendre ODE with order m and its solution is of the form

$$\phi(r)=c_1P_m^n(r)+c_2Q_m^n(r);\qquad m=k-\tfrac{1}{2}\qquad n(\text{order})=\tfrac{1}{2}(1+4b/a)^{1/2}.$$

This implies that

$$w(x)=\ln\left\{(c_1P_m^n(\coth x)+c_2Q_m^n(\coth x))(\sinh x)^{-1/2}\right\}$$

Thus the solution takes a form

$$u(x,y,t)=\begin{cases} \dfrac{e^{(n^2-1/4)t}}{(\sinh x)^{1/2}}(k_2P_m^n(\coth x)+k_3Q_m^n(\coth x))\cos(ky+k_1); c<0 \\ \dfrac{e^{(n^2-1/4)t}}{(\sinh x)^{1/2}}(k_2P_{-1/2}^n(\coth x)+k_3Q_{-1/2}^n(\coth x))(y+k_1); \quad c=0 \end{cases}$$

• *Hyperboloid of one sheet* $f(x)=\ln\cosh(x)$

The Eq(6.6) takes a form

$$w''+w'^2+w'\tanh x+c\operatorname{sech}^2 x-b/a=0$$

Putting $w(x)=\ln(\phi(x)(\cosh x)^{-1/2})$ reduces the above equation to the form

$$\phi''(x)-((c-\tfrac{1}{4})\tanh^2 x-c+\tfrac{1}{2}+\tfrac{b}{a})\varphi(x)=0$$

Let $r = \tanh x$, this implies that

$$(1-r^2)\phi''(r) - 2r\phi'(r) + \left\{c - \tfrac{1}{4} - \tfrac{a+4b}{4a(1-r^2)}\right\}\phi(r) = 0$$

For $c \geq 0,\ c = k^2,$ we have an associated Legendre ODE with order m and its solution is of the form

$$\phi(r) = c_1 P_m^n(r) + c_2 Q_m^n(r); \qquad m = k - \tfrac{1}{2} \qquad n(\text{order}) = \tfrac{1}{2}(1 + 4b/a)^{1/2}.$$

This implies that

$$w(x) = \ln\left\{(c_1 P_m^n(\tanh x) + c_2 Q_m^n(\tanh x))(\cosh x)^{-1/2}\right\}$$

Thus the solution takes a form

$$u(x,y,t) = \begin{cases} \dfrac{e^{(n^2-1/4)t}}{(\cosh x)^{1/2}}(k_2 P_m^n(\tanh x) + k_3 Q_m^n(\tanh x))\cos(ky + k_1); c > 0 \\ \dfrac{e^{(n^2-1/4)t}}{(\cosh x)^{1/2}}(k_2 P_{-1/2}^n(\tanh x) + k_3 Q_{-1/2}^n(\tanh x))(y + k_1); \quad c = 0 \end{cases}$$

• *Pseudosphere* or *tractoid* $(f(x) = x)$.

The Eq(6.6) takes a form

$$w' + w'^2 + w'' + ce^{-2x} - b/a = 0$$

Letting $w(x) = \ln(\phi(x))$ gives

$$\phi''(x) + \phi'(x) + (ce^{-2x} - b/a)\phi(x) = 0$$

For $c > 0,\ c = k^2,\ k > 0$ we let $r = ke^{-x}$, this gives

$$r^2\phi''(r) + \phi(r)(r^2 - b/a) = 0$$

Letting $\phi(r) = \varphi(r)\sqrt{r}$ gives a Bessel's differential equation below

$$r^2\varphi''(r) + r\varphi'(r) + \varphi(r)(r^2 - m^2) = 0;\ m^2 = (\tfrac{1}{4} + b/a), \quad b/a \geq -\tfrac{1}{4}$$

Whose solution is of the form

$$\varphi(r) = c_1 J_m(r) + c_2 Y_m(r)$$

$$w(x) = \ln\left\{e^{-x/2}(c_1 J_m(ke^{-x}) + c_2 Y_m(ke^{-x}))\right\}$$

Where J_m and Y_m are first and second kind Bessel functions of order m. the solution in case takes a form

$$u(x,y,t) = e^{(m^2-1/4)t-x/2}(k_3 J_m(ke^{-x}) + k_4 Y_m(ke^{-x}))\cosh(ky + k_1)$$

For $c = 0$, we have a linear second order ODE

$$\phi''(x) + \phi'(x) - \phi(x)b / a = 0$$

whose solution takes a form

$$\phi(x) = \begin{cases} c_1 e^{-(n-1)x/2} + c_2 e^{-(n+1)x/2}; & 1 + 4b / a > 0, \quad 1 + 4b / a = n^2 \\ c_1 \mathrm{e}^{-x/2} \sin(\frac{nx}{2}) + c_2 \mathrm{e}^{-x/2} \cos(\frac{nx}{2}); & 1 + 4b / a < 0, \ 1 + 4b / a = -n^2 \end{cases}$$

This implies that

$$w(x) = \begin{cases} \ln\left\{c_1 e^{(n-1)x/2} + c_2 e^{-(n+1)x/2}\right\}; & 1 + 4b / a > 0, \quad 1 + 4b / a = n^2 \\ \ln\left\{c_1 e^{-x/2} \sin(\frac{nx}{2}) + c_2 e^{-x/2} \cos(\frac{nx}{2})\right\}; & 1 + 4b / a < 0, \ 1 + 4b / a = -n^2 \end{cases}$$

Thus the solution takes the form

$$u(x, y, t) = \begin{cases} e^{(n^2-1)/4}(k_3 e^{-(n-1)x/2} + k_4 e^{-(n+1)x/2})(y + k_1); & 1 + 4b / a = n^2 \\ e^{-(n^2+1)/4}(k_3 e^{-x/2} \sin(\frac{nx}{2}) + k_4 e^{-x/2} \cos(\frac{nx}{2}))(y + k_1); & 1 + 4b / a = -n^2 \end{cases}$$

6.1.2. Reduction by 2-dimensional subalgebra.

In this section, we use 2-dimensional subalgebra to reduce the number of independent variables of the Eq(1.1)

I. Subalgebra $\mathcal{L} = \langle X_1, aX_2 + bX_3 \rangle$

$$X_1 = \frac{\partial}{\partial y}$$

The characteristic system of

$$X_1 \mathcal{I} = 0$$

is given by

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{1} = \frac{du}{0}$$

Solving the characteristic system gives the similarity variables below.

$$\xi_1 = x, \quad \xi_2 = t \ \text{ and } \ V(\xi_1, \xi_2) = u\,.$$

Substitution of the similarity variables in Eq(1.1) and using the chain rule implies the solution of Eq(1.1) is of the form $u = V(\xi_1, \xi_2)$.

We now observe that

$$u_x = V_{\xi_1},\ u_{xx} = V_{\xi_1\xi_1},\ u_y = 0,\ u_{yy} = 0 \text{ and } u_t = V_{\xi_2}.$$

Clearly $V(\xi_1, \xi_2)$ satisfies reduced PDE in 2 independent variables.

$$V_{\xi_2} = f'(\xi_1)V_{\xi_1} + V_{\xi_1\xi_1} \tag{6.8}$$

We now note that

$$[X_1, aX_2 + bX_3] = 0$$

Therefore the two symmetries commute.

It then follows immediately from theorem on P-285 of [23] that the second symmetry is inherited by PDE (6.8), as it commutes with the first symmetry hence

$$Y = a\frac{\partial}{\partial \xi_2} + bV\frac{\partial}{\partial V}$$

is a symmetry of the PDE(6.8)

The characteristic system of

$$Y\mathcal{I} = 0$$

is given by

$$\frac{d\xi_1}{0} = \frac{d\xi_2}{a} = \frac{dV}{bV}$$

Solving the characteristic system gives the similarity variables below.

$$r(\xi_1, \xi_2) = \xi_1 \text{ and } w(r) = \ln V - \frac{b}{a}\xi_2. .$$

Substitution of the similarity variables in Eq(6.8) and using the chain rule implies the solution of Eq(1.8) is of the form $V = e^w e^{\frac{b}{a}\xi_2}$.

$$V_{\xi_1} = w_r e^w e^{\frac{b}{a}\xi_2},\ \ V_{\xi_1\xi_1} = (w_{rr} + w_r^2)_r e^w e^{\frac{b}{a}\xi_2} \text{ and } V_{\xi_2} = \frac{b}{a}e^w e^{\frac{b}{a}\xi_2}.$$

This reduces the Eq(1.8) to an ODE

$$\frac{d^2w}{dr^2} + \left(\frac{dw}{dr}\right)^2 + f'(r)\frac{dw}{dr} = \frac{b}{a} \tag{6.9}$$

Putting $w(r) = \ln(\phi(r))$ reduces the Eq(6.9) to a second order linear ODE

$$\phi''(r) + f'(r)\phi'(r) - c\phi(r) = 0 \qquad c = b / a \tag{6.10}$$

By back substitution of the similarity variables we obtain the solution

$$u(x,y,t) = \phi(x)e^{ct}. \tag{6.11}$$

Next we obtain the solution for some common surfaces of revolution.

- *Cylinder of a unit radius* $(f(x) = 0)$

The Eq(6.10) takes a form

$$\phi''(r) - c\phi(r) = 0$$

This implies that

$$\phi(r) = \begin{cases} k_1 \sin(kr) + k_2 \cos(kr), & c = -k^2 \\ k_1 e^{-kr} + k_2 e^{kr}, & c = k^2 \end{cases}$$

thus the solution (6.11) takes a form

$$u(x,y,t) = \begin{cases} e^{-k^2 t}(k_1 \sin(kx) + k_2 \cos(kx)), & c = -k^2 \\ e^{k^2 t}(k_1 e^{-kx} + k_2 e^{kx}), & c = k^2 \end{cases}$$

- *Cone* $(f(x) = \ln lx;\ x > 0,\ l > 0,\ l \neq 0)$

The Eq(6.10) takes a form

$$r\phi''(r) + \phi'(r) - cr\phi(r) = 0$$

This implies that

$$\phi(r) = \begin{cases} k_1 J_0(kr) + k_2 Y_0(kr), & c = -k^2 \\ k_1 I_0(kr) + k_2 K_0(kr), & c = k^2 \end{cases}$$

where J_0 and Y_0 are first and second kind Bessel functions were as I_0 and K_0 first and second kind modified Bessel functions.

Therefore the solution (6.11) takes a form

$$u(x,y,t) = \begin{cases} e^{-k^2 t}(k_1 J_0(kr) + k_2 Y_0(kr)), & c = -k^2 \\ e^{k^2 t}(k_1 I_0(kr) + k_2 K_0(kr)) & c = k^2 \end{cases}$$

- *Unit sphere* $(f(x) = \ln\cos x; -0.5\pi < x < 0.5\pi$

The Eq(6.10) takes a form

$$\phi''(r) - \tan(r)\phi'(r) - c\phi(r) = 0$$

Putting $z = \sin(r)$ transforms the above equation into a Legendre ODE

$$(1 - z^2)\phi''(z) - 2z\phi'(z) - c\varphi(z) = 0$$

whose solution is of the form.

$$\phi(z) = k_1 P_m(z) + k_2 Q_m(z), \quad m(\text{degree}) = \tfrac{1}{2}\left\{(1-4c)^{1/2} - 1\right\},\ 1 - 4c > 0.$$

P and Q are Legendre functions of the first and second kinds

This implies that

$$\phi(r) = k_1 P_m(\sin r) + k_2 Q_m(\sin r)$$

P and Q are Legendre associated functions of the first and second kinds.

Thus the solution (6.11) takes a form

$$u(x, y, t) = e^{-m(m+1)t}(k_1 P_m(\sin r) + k_2 Q_m(\sin r))$$

• *Surface of a conic type with* $f(x) = \ln\sinh(x);\ x > 0$

The Eq(6.10) takes a form

$$\phi''(r) + \coth(r)\phi'(r) - c\phi(r) = 0$$

Putting $\phi(r) = \varphi(r)(\sinh r)^{-1/2}$ reduces the above equation to the form

$$\varphi''(r) - \tfrac{1}{4}(4c + \coth^2 r - 2\operatorname{csch}^2 r) = 0$$

Let $z = \coth r$, this implies that

$$(1 - z^2)\varphi''(z) - 2z\varphi'(z) + \left\{-\tfrac{1}{4} - \tfrac{(1+4c)/4}{1-z^2}\right\}\phi(z) = 0$$

This is an associated Legendre ODE whose solution is of the form.

$$\phi(z) = k_1 P_{-1/2}^{m}(z) + k_2 Q_{-1/2}^{m}(z);\ \text{degree} = -1/2,\ m(\text{order}) = \tfrac{1}{2}(1+4c)^{1/2}.$$

P and Q are Legendre associated functions of the first and second kinds.

This implies that the solution (6.11) is of the form

$$u(x, y, t) = e^{(m^2 - 1/4)t}\left\{(k_1 P_{-1/2}^{m}(\coth x) + k_2 Q_{-1/2}^{m}(\coth x))(\sinh x)^{-1/2}\right\}$$

• *Hyperboloid of one sheet* $f(x) = \ln\cosh(x)$

The Eq(6.10) takes a form

$$\phi''(r) + \coth(r)\phi'(r) - c\phi(r) = 0$$

Putting $w(x) = \ln(\phi(x)(\cosh x)^{-1/2})$ reduces the above equation to the form

$$\phi''(r) - \tfrac{1}{4}(1 + \operatorname{sech}^2 r + 4c)\varphi(r) = 0$$

Let $z = \tanh r$, this implies that

$$(1 - z^2)\varphi''(z) - 2z\varphi'(z) + (-\tfrac{1}{4} - \tfrac{(1+4c)/4}{1-z^2})\phi(z) = 0$$

This is an associated Legendre ODE whose solution is of the form.

$$\phi(z) = k_1 P^m_{-1/2}(z) + k_2 Q^m_{-1/2}(z);\ \text{degree} = -1/2,\ m(\text{order}) = \tfrac{1}{2}(1 + 4c)^{1/2}.$$

P and Q are Legendre associated functions of the first and second kinds.
By (6.11) we note that

$$u(x, y, t) = e^{(m^2 - 1/4)t}\left\{(k_1 P^m_{-1/2}(\tanh x) + k_2 Q^m_{-1/2}(\tanh x))(\cosh x)^{-1/2}\right\}$$

• *Pseudosphere* or *tractoid* $(f(x) = x)$.

The Eq(1.10) takes a form

$$\phi''(r) + \phi'(r) - c\phi(r) = 0$$

This turns out to be linear second ODE whose solution is given by

$$\phi(r) = \begin{cases} e^{-r/2}(k_1 \sin kr + k_2 \cos kr), & 1/4 + c < 0,\ 1/4 + c = -k^2 \\ e^{-r/2}(k_1 r + k_2), & 1/4 + c = 0 \\ e^{-r/2}(k_1 e^{kr} + k_2 e^{-kr}), & 1/4 + c > 0,\quad 1/4 + c = k^2 \end{cases}$$

This implies that the solution (6.11) is of the form

$$u(x, y, t) = \begin{cases} e^{ct - x/2}(k_1 \sin kx + k_2 \cos kx), & 1/4 + c < 0,\ 1/4 + c = -k^2 \\ e^{ct - x/2}(k_1 x + k_2), & 1/4 + c = 0 \\ e^{ct - x/2}(k_1 e^{kx} + k_2 e^{-kx}), & 1/4 + c > 0,\quad 1/4 + c = k^2 \end{cases}$$

• *Torus* $(f(x) = \ln(1 + \cos x);\ x \in [0, \pi) \cup (\pi, 2\pi)$

The Eq(6.10) takes a form

$$\phi''(r) - \tan(r/2)\phi'(r) - c\phi(r) = 0 \qquad c = b/a$$

The substitution $\phi(r) = \varphi(r)\sec(r/2)$ reduces the above ODE to

$$4\varphi''(r) - (4c - 1)\varphi(r) = 0$$

The solution the above ODE is

$$\varphi(r)=\begin{cases} k_1e^{mr/2}+k_2e^{-mr/2}; & m^2=4c-1\\ k_1r+k_2; & 4c-1=0\\ k_1\cos(mr/2)+k_2\sin(mr/2); & m^2=1-4c\end{cases}$$

Hence

$$\phi(r)=\begin{cases} (k_1e^{mr/2}+k_2e^{-mr/2})\sec(r/2); & m^2=4c-1\\ (k_1r+k_2)\sec(r/2); & 4c-1=0\\ [k_1\cos(mr/2)+k_2\sin(mr/2)]\sec(r/2); & m^2=1-4c\end{cases}$$

By (6.11) we note that the solution is

$$u(x,y,t)=\begin{cases} e^{ct}(k_1e^{mr/2}+k_2e^{-mr/2})\sec(r/2); & m^2=4c-1\\ e^{ct}(k_1r+k_2)\sec(r/2); & 4c-1=0\\ e^{ct}[k_1\cos(mr/2)+k_2\sin(mr/2)]\sec(r/2); & m^2=1-4c\end{cases}$$

6.2. Symmetry reductions and exact solutions of the wave equation on surface of revolution.

In chapter 5 we discussed the minimal symmetry algebra for the wave equation

$$u_{tt}=f'(x)u_x+u_{xx}+e^{-2f(x)}u_{yy} \tag{1.2}$$

on surface of revolution and we noted that it is spanned by the vector fields

$$X_1=\frac{\partial}{\partial y},\quad X_2=\frac{\partial}{\partial t},\quad X_3=u\frac{\partial}{\partial u}$$

6.2.1. Reduction by 1-dimensional subalgebra.

Using one of the cases of 1–dimensional subalgebra, we reduce the Eq(1.2) to a PDE with two independent variables. The reduced PDE is then tried for solution for an arbitrary determining function f using other methods.

I. Subalgebra $\mathcal{L}=\langle X_1+aX_2\rangle,\ a\neq 0$

$$X = \frac{\partial}{\partial y} + a\frac{\partial}{\partial t}$$

The characteristic system of

$$X\mathcal{I} = 0$$

is given by

$$\frac{dt}{a} = \frac{dx}{0} = \frac{dy}{1} = \frac{du}{0}$$

Solving the characteristic system gives the similarity variables below.

$$\xi_1 = x, \quad \xi_2 = y - \frac{t}{a} \quad \text{and} \quad V(\xi_1, \xi_2) = u.$$

Substitution of the similarity variables in Eq(1.2) and using the chain rule implies the solution of Eq(1.2) is of the form $u = V(\xi_1, \xi_2)$.

We observe that

$$u_x = V_{\xi_1}, \; u_{xx} = V_{\xi_1\xi_1}, \; u_y = V_{\xi_2}, \; u_{yy} = V_{\xi_2\xi_2}, \; u_t = -\frac{1}{a}V_{\xi_2} \; \text{and} \; u_{tt} = \frac{1}{a^2}V_{\xi_2\xi_2}.$$

Clearly $V(\xi_1, \xi_2)$ satisfies the reduced PDE in 2 independent variables.

$$\frac{1}{a^2}V_{\xi_2\xi_2} = f'(\xi_1)V_{\xi_1} + V_{\xi_1\xi_1} + e^{-2f(\xi_1)}V_{\xi_2\xi_2} \tag{6.12}$$

a. If we let $V(\xi_1, \xi_2) = v(\xi_1) + w(\xi_2)$, then $v(\xi_1)$ and $w(\xi_2)$ respectively satisfy the reduced ODEs below.

$$v'' + f'(\xi_1)v' = (a^{-2} - e^{-2f(\xi_1)})c, \tag{6.13}$$

$$w'' = c \tag{6.14}$$

For the Eq(6.13), we note that by the method of integrating factor

$$\frac{dv}{d\xi_1} = (ca^{-2}\int(e^{f(\xi_1)} - e^{-f(\xi_1)}a^2)d\xi_1 + k_1)e^{-f(\xi_1)}$$

This implies that

$$v(\xi_1) = \int\left\{(ca^{-2}\int(e^{f(\xi_1)} - e^{-f(\xi_1)}a^2)d\xi_1 + k_1)e^{-f(\xi_1)}\right\}d\xi_1 + c_1$$

Were as for the Eq(6.14) the solution takes the form

$$w(\xi_2) = \tfrac{1}{2}c\xi_2^2 + c_2\xi_2 + c_3$$

Thus by back substitution we obtain an invariant solution.

$$u(x,y,t)=\int\left\{(ca^{-2}\int(e^{f(x)}-e^{-f(x)}a^{2})dx+k_{1})e^{-f(x)}\right\}dx+\tfrac{1}{2}ca^{-2}(ay-t+k_{2})^{2}+k_{3}$$

Solutions of the wave equation on some known surfaces of revolution

- *Cylinder of a unit radius* $(f(x)=0)$

$$u(x,y,t)=\tfrac{1}{2}ca^{-2}(x^{2}+(ay-t+k_{2})^{2})+k_{1}x+k_{3}$$

- *Cone* $(f(x)=\ln lx;\ \ x>0,\ \ l>0,\ \ l\neq 0)$

$$u(x,y,t)=\tfrac{1}{4}ca^{-2}(x^{2}+2ca^{-2}(ay-t+k_{2})^{2})-\tfrac{1}{2}l^{-2}(c\ln x-2k_{1}l)\ln x+k_{3}$$

- *Unit sphere* $(f(x)=\ln\cos x;-0.5\pi<x<0.5\pi$

$$u(x,y,t)=(k_{1}-\tfrac{1}{2}c\ln(\sec x+\tan x))\ln(\sec x+\tan x)$$
$$+\tfrac{1}{2}ca^{-2}((ay-t+k_{2})^{2}-\ln\cos^{2}x)+k_{3}$$

- *Surface of a conic type with* $f(x)=\ln\sinh x;\ \ x>0$

$$u(x,y,t)=\tfrac{1}{2}ca^{-2}(\ln\sinh^{2}x+(ay-t+k_{2})^{2})-(c\tanh^{-1}e^{x}+2k_{1})\tanh^{-1}e^{x}+k_{3}$$

- *Hyperboloid of one sheet* $f(x)=\ln\cosh x;\ \ x>0$

$$u(x,y,t)=\tfrac{1}{2}ca^{-2}(\ln\cosh^{2}x+(ay-t+k_{2})^{2})-2(c\tan^{-1}e^{x}-k_{1})\tan^{-1}e^{x}+k_{3}$$

- *Pseudosphere* or *tractoid* $(f(x)=x)$.

$$u(x,y,t)=\tfrac{1}{2}ca^{-2}(2x+(ay-t+k_{2})^{2})-\tfrac{1}{2}ce^{2x}-k_{1}e^{-x}+k_{3}$$

- *Paraboloid* $f(x)=\tfrac{1}{2}\ln x,\ \ x>0.$

$$u(x,y,t)=\tfrac{1}{6}ca^{-2}(3(ay-t+k_{2})^{2}+2x^{2})+2(k_{1}x^{1/2}-cx)+k_{3}$$

- *Torus* $(f(x)=\ln(1+\cos x);\ \ x\in[0,\pi)\cup(0,2\pi)$

$$u(x,y,t)=\tfrac{1}{a^{2}}cx\tan(\tfrac{1}{2}x)-\tfrac{1}{2}c\sec^{2}(\tfrac{1}{2}x)+k_{1}\tan(\tfrac{1}{2}x)+\tfrac{1}{2}ca^{-2}(ay-t+k_{2})^{2}+k_{3}$$

b. If we let $V(\xi_{1},\xi_{2})=v(\xi_{1})w(\xi_{2})$, then $v(\xi_{1})$ and $w(\xi_{2})$ respectively satisfy the reduced ODEs below.

$$v''+f'(\xi_{1})v'=c(a^{-2}-e^{-2f(\xi_{1})})v \tag{6.15}$$

$$w'' - cw = 0 \tag{6.16}$$

For $c = 0$, it can easily shown that

$$v(\xi_1) = k_1 + k_2 \int e^{-f(\xi_1)} d\xi_1 \quad \text{and} \quad w(\xi_2) = k_3 + k_4 \xi_2$$

This implies that the invariant solution is of the form

$$u(x, y, t) = (k_3 + k_4(ay - t))(k_1 + k_2 \int e^{-f(x)} dx)$$

For $c \neq 0$ from Eq(6.16)

$$w'' - cw = 0$$

we note that

$$w(\xi_2) = \begin{cases} c_1 \sin(k\xi_2) + c_2 \cos(k\xi_2); & c < 0, \quad c = -k^2 \\ c_1 e^{k\xi_2} + c_2 e^{-k\xi_2}; & c > 0, \quad c = k^2 \end{cases}$$

Thus by back substitution, the invariant solution takes the form

$$u(x, y, t) = \begin{cases} v(x)\left\{c_1 \sin(k(y - t/a)) + c_2 \cos(k(y - t/a))\right\}; & c = -k^2 \\ v(x)\left\{c_1 e^{k(y-t/a)} + c_2 e^{-k(y-t/a)}\right\}; & c = k^2 \end{cases}$$

where $v(x)$ satisfies the equation

$$v'' + f'(x)v' = c(a^{-2} - e^{-2f(x)})v$$

Next we obtain the solution for some common surfaces of revolution.

- *Cylinder of a unit radius* $(f(x) = 0)$

$$v'' - c(a^{-2} - 1)v = 0$$

For $a^{-2} - 1 < 0$, we let $a^{-2} - 1 = -m^2$, $m > 0$, therefore

$$u(x, y, t) = \begin{cases} \left\{k_3 e^{mkx} + k_4 e^{-mkx}\right\}\left\{k_1 \sin(k(y - t/a)) + k_2 \cos(k(y - t/a))\right\}; & c = -k^2 \\ \left\{k_3 \sin(mkx) + k_4 \cos(mkx)\right\}\left\{k_1 e^{k(y-t/a)} + k_2 e^{-k(y-t/a)}\right\}; & c = k^2 \end{cases}$$

For $a^{-2} - 1 = 0$, we have

$$u(x, y, t) = \begin{cases} \left\{k_3 x + k_4\right\}\left\{k_1 \sin(k(y \pm t)) + k_2 \cos(k(y \pm t))\right\}; & c = -k^2 \\ \left\{k_3 x + k_4\right\}\left\{k_1 e^{k(y\pm t)} + k_2 e^{-k(y\pm t)}\right\}; & c = k^2 \end{cases}$$

For $a^{-2}-1>0$, we let $a^{-2}-1=m^2,\ m>0$

Therefore

$$u(x,y,t)=\begin{cases}\left(k_3\sin(mkx)+k_4\cos(mkx)\right)\left(k_1\sin(k(y-t/a))+k_2\cos(k(y-t/a))\right); & c=-k^2\\ k_3e^{mxa}+k_4e^{-mkx}\left(k_1e^{k(y-t/a)}+k_2e^{-k(y-t/a)}\right); & c=k^2\end{cases}$$

- *Cone* $(f(x)=\ln lx;\ x>0,\ l>0)$

The Eq(6.15) takes a form

$$x^2v''+xv'-c\left(\frac{x^2}{a^2}-\frac{1}{l^2}\right)v=0$$

For $c<0$ i.e. $c=-k^2,\ k>0$ we have Bessel ODE whose solution is of the form

$$v(x)=k_1J_{k/l}\left(\frac{kx}{a}\right)+k_2Y_{k/l}\left(\frac{kx}{a}\right)$$

where $J_{k/l}$ and $Y_{k/l}$ are first and second kind Bessel functions

This implies that the invariant solution is given by

$$u(x,y,t)=\left(k_1\sin(k(y-t/a))+k_2\cos(k(y-t/a))\right)\left(k_3J_{k/l}(kx/a)+k_3Y_{k/l}(kx/a)\right)$$

- *Unit sphere* $(f(x)=\ln\cos x;-0.5\pi<x<0.5\pi$

Eq(6.15) takes a form

$$v''-\tan(x)v'-c(a^{-2}-\sec^2x)v=0$$

Putting $z=\sin(x)$ transforms the above equation into a Legendre associated ODE if $c<0$ i.e. $c=-k^2,\ k>0$

$$(1-z^2)v''(z)-2uv'(z)+\left(\frac{k^2}{a^2}-\frac{k^2}{1-z^2}\right)v(z)=0$$

whose solution is of the form.

$$v(z)=k_3P_m^k(z)+k_4Q_m^k(z);\quad m(\text{degree})=\tfrac{1}{2a}((a^2+4k^2)^{\frac{1}{2}}-a),\quad k-order.$$

P and Q are associated Legendre functions of the first and second kinds

Thus the solution takes a form

$$u(x,y,t)=\left(k_1\sin(k(y-t/a))+k_2\cos(k(y-t/a))\right)\left(k_3P_m^k(\sin x)+k_4Q_m^k(\sin x))\right)$$

• *Surface of a conic type with* $f(x) = \ln\sinh(x);\ x > 0$

The Eq(6.15) takes a form

$$v'' + \coth(x)v' - c(a^{-2} - \operatorname{csch}^2 x)v = 0$$

Putting $v(x) = \phi(x)(\sinh x)^{-1/2}$ reduces the above equation to the form

$$\phi''(x) + \tfrac{1}{4}\phi(x)(\coth^2 x - 2) - c(a^{-2} - \operatorname{csch}^2 x)\phi(x) = 0$$

For $c < 0$, $c = -k^2,\ k > 0$ letting $z = \coth x$, implies that

$$(1-z^2)\phi''(z) - 2z\phi'(z) + \left(k^2 - \frac{1}{4} + \frac{(4k^2 - a^2)/4a^2}{(1-z^2)}\right)\varphi(z) = 0$$

This is a Legendre associated ODE whose solution is of the form.

$$\phi(z) = k_3 P_m^n(z) + k_4 Q_m^n(z); \quad m(\text{degree}) = k - 1/2, \quad n(\text{order}) = \frac{1}{2a}\sqrt{a^2 - 4k^2}.$$

P and Q are Legendre associated functions of the first and second kinds. This implies that

$$u(x,y,t) = \frac{\left(k_1 \sin(k(y - t/a)) + k_2 \cos(k(y - t/a))\right)\left(k_3 P_m^n(\coth x) + k_4 Q_m^n(\coth x)\right)}{(\sinh x)^{1/2}}$$

• *Hyperboloid of one sheet* $f(x) = \ln\cosh(x)$

The Eq(6.15) takes a form

$$v'' + v'\tanh x - c(a^{-2} - \operatorname{sech}^2 x)v = 0$$

If $c > 0$, $c = k^2$ then the substitution of $z = \tanh x$ and $v(x) = \ln(\phi(z)(\cosh x)^{-1/2})$ transforms the above equation into a Legendre associated ODE

$$(1-z^2)\phi''(z) - 2z\phi'(z) + \left(k^2 - \tfrac{1}{4} - \frac{k^2 + \tfrac{1}{4}a^2}{a^2(1-z^2)}\right)\varphi(z) = 0$$

whose solution is given by

$$\phi(z) = k_1 P_m^n(z) + k_2 Q_m^n(z);\ m(\text{degree}) = k - \tfrac{1}{2},\ n(\text{order}) = \frac{(4k^2 + a^2)^{1/2}}{2a}.$$

This implies that

$$u(x,y,t) = \left(k_3 P_m^n(z) + k_4 Q_m^n(z)\right)\left(k_1 e^{k(y - t/a)} + k_2 e^{-k(y-t/a)}\right)$$

• *Pseudosphere* or *tractoid* $(f(x) = x)$.

The Eq(6.15) takes a form

$$v'' + v' - c(a^{-2} - e^{-2x})v = 0$$

The change of variable $z = e^{-x}$ and $v(x) = \phi(z)e^{-x/2}$ gives

$$z^2\phi''(z) + z\phi'(z) + (cz^2 - ca^{-2} - \tfrac{1}{4})\varphi(z) = 0$$

For $c < 0$, i.e. $c = -k^2$ we have modified Bessel ODE were as for $c > 0$, i.e. $c = k^2$ we have Bessel ODE. This implies that

$$\phi(z) = \begin{cases} k_3 I_m(kz) + k_4 K_m(kz), & c = -k^2,\ m = \frac{1}{2a}(a^2 - 4k^2)^{1/2} \\ k_3 J_n(kz) + k_4 Y_n(kz), & c = k^2, \quad n = \frac{1}{2a}(a^2 + 4k^2)^{1/2} \end{cases}$$

Thus the invariant solution takes the form

$$u(x,y,t) = \begin{cases} e^{-x/2}\left(k_3 I_m(ke^{-x}) + k_4 K_m(ke^{-x})\right)\left(c_1 \sin(k(y - t/a)) + c_2 \cos(k(y - t/a))\right); & c = -k^2 \\ e^{-x/2}\left(k_3 J_n(ke^{-x}) + k_4 Y_n(ke^{-x})\right)\left(c_1 e^{k(y-t/a)} + c_2 e^{-k(y-t/a)}\right); & c = k^2 \end{cases}$$

6.2.2. Reduction by 2-dimensional subalgebra

In this section, we use 2-dimensional subalgebra to reduce the number of independent variables of the Eq(1.2). We only consider one of the cases of 2- dimensional subalgebra.

Subalgebra $\mathcal{L} = \langle X_1, aX_2 + bX_3 \rangle$

$$X_1 = \frac{\partial}{\partial y}$$

The characteristic system of

$$X_1 \mathcal{I} = 0$$

is given by

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{1} = \frac{du}{0}$$

Solving the characteristic system gives the similarity variables below.

$$\xi_1 = x, \quad \xi_2 = t \ \text{ and } \ V(\xi_1, \xi_2) = u\,.$$

Substitution of the similarity variables in Eq(2.1) and using the chain rule implies the solution of Eq(2.1) is of the form $u = V(\xi_1, \xi_2)$.

We now observe that

$$u_x = V_{\xi_1},\ u_{xx} = V_{\xi_1\xi_1},\ u_y = 0,\ u_{yy} = 0, u_t = V_{\xi_2} \text{ and } u_{tt} = V_{\xi_2\xi_2}.$$

Clearly $V(\xi_1, \xi_2)$ satisfies reduced PDE in 2 independent variables.

$$V_{\xi_2\xi_2} = f'(\xi_1)V_{\xi_1} + V_{\xi_1\xi_1} \tag{6.17}$$

We now note that

$$[X_1, aX_2 + bX_3] = 0$$

Therefore the two symmetries commute.

It then follows immediately from theorem on P-285 of [23] that the second symmetry is inherited by PDE (6.17), as it commutes with the first symmetry hence

$$Y = a\frac{\partial}{\partial \xi_2} + bV\frac{\partial}{\partial V}$$

is a symmetry of the PDE(6.17)

The characteristic system of

$$Y\mathcal{I} = 0$$

is given by

$$\frac{d\xi_1}{0} = \frac{d\xi_2}{a} = \frac{dV}{bV}$$

Solving the characteristic system gives the similarity variables below.

$$r(\xi_1, \xi_2) = \xi_1 \text{ and } w(r) = \ln V - \frac{b}{a}\xi_2. \,.$$

Substitution of the similarity variables in Eq(2.7) and using the chain rule implies the solution of Eq(2.7) is of the form $V = e^w e^{\frac{b}{a}\xi_2}$.

$$V_{\xi_1} = w_r e^w e^{\frac{b}{a}\xi_2},\ \ V_{\xi_1\xi_1} = (w_{rr} + w_r^2)_r e^w e^{\frac{b}{a}\xi_2},\ \ V_{\xi_2} = \frac{b}{a}e^w e^{\frac{b}{a}\xi_2},\ \ V_{\xi_2\xi_2} = \frac{b^2}{a^2}e^w e^{\frac{b}{a}\xi_2}.$$

This reduces the Eq(6.17) to an ODE

$$\frac{d^2w}{dr^2} + \left(\frac{dw}{dr}\right)^2 + f'(r)\frac{dw}{dr} = \frac{b^2}{a^2} \tag{6.18}$$

Putting $w(r) = \ln(\phi(r))$ reduces the Eq(6.18) to a second order linear ODE

$$\phi''(r) + f'(r)\phi'(r) - k^2\phi(r) = 0 \qquad k = b / a \tag{6.19}$$

By back substitution of the similarity variable we obtain the solution

$$u(x,y,t) = \phi(x)e^{kt}. \tag{6.20}$$

Next we obtain the solution for some common surfaces of revolution.

- *Cylinder of a unit radius* $(f(x) = 0)$

$$\phi''(r) - k^2\phi(r) = 0$$

This implies that

$$\phi(r) = k_1 e^{-kr} + k_2 e^{kr}$$

thus the solution takes a form

$$u(x,y,t) = e^{kt}(k_1 e^{-kx} + k_2 e^{kx})$$

- *Cone* $(f(x) = \ln lx;\ x > 0,\ l > 0)$

The Eq(2.9) takes a form

$$r\phi''(r) + \phi'(r) - k^2 r\phi(r) = 0$$

This implies that

$$\phi(r) = k_1 I_0(kr) + k_2 K_0(kr)$$

where J_0 and Y_0 are first and second kind Bessel functions. Therefore the solution (6.20) takes a form

$$u(x,y,t) = e^{kt}(k_1 I_0(kr) + k_2 K_0(kr))$$

- *Unit sphere* $(f(x) = \ln\cos x; -0.5\pi < x < 0.5\pi$

The Eq(6.19) takes a form

$$\phi''(r) - \tan(r)\phi'(r) - k^2\phi(r) = 0$$

Putting $z = \sin(r)$ transforms the above equation into a Legendre ODE

$$(1 - z^2)\phi''(z) - 2z\phi'(z) - k^2\varphi(z) = 0$$

whose solution is of the form.

$$\varphi(z) = k_1 P_m(z) + k_2 Q_m(z), \quad m(\text{degree}) = \tfrac{1}{2}\left\{(1 - 4k^2)^{1/2} - 1\right\},\ 1 - 4k^2 > 0.$$

P and Q are Legendre functions of the first and second kinds

This implies that

$$\varphi(r) = k_1 P_m(\sin r) + k_2 Q_m(\sin r)$$

Thus the solution takes a form

$$u(x, y, t) = e^{-m(m+1)t}(k_1 P_m(\sin r) + k_2 Q_m(\sin r))$$

• *Surface of a conic type with* $f(x) = \ln\sinh(x);\ x > 0$

The Eq(6.19) takes a form

$$\phi''(r) + \coth(r)\phi'(r) - k^2\phi(r) = 0$$

Putting $\phi(r) = \varphi(r)(\sinh r)^{-1/2}$ reduces the above equation to the form

$$\varphi''(r) - \tfrac{1}{4}(4k^2 + \coth^2 r - 2\operatorname{csch}^2 r) = 0$$

Let $z = \coth r$, this implies that

$$(1 - z^2)\varphi''(z) - 2z\varphi'(z) + \left\{-\tfrac{1}{4} - \tfrac{(1+4k^2)/4}{1-z^2}\right\}\phi(z) = 0$$

This is an associated Legendre ODE with order m and its solution is of the form.

$$\phi(z) = k_1 P^m_{-1/2}(z) + k_2 Q^m_{-1/2}(z);\ \text{degree} = -1/2,\ m(\text{order}) = \tfrac{1}{2}(1 + 4k^2)^{1/2}.$$

This implies that

$$u(x, y, t) = e^{(m^2-4)^{-1/2}t/4}\left\{(k_1 P^m_{-1/2}(\coth x) + k_2 Q^m_{-1/2}(\coth x))(\sinh x)^{-1/2}\right\}$$

• *Hyperboloid of one sheet* $f(x) = \ln\cosh(x)$

The Eq(6.19) takes a form

$$\phi''(r) + \coth(r)\phi'(r) - k^2\phi(r) = 0$$

Putting $w(x) = \ln(\phi(x)(\cosh x)^{-1/2})$ reduces the above equation to the form

$$\phi''(r) - \tfrac{1}{4}(1 + \operatorname{sech}^2 r + 4k^2)\varphi(r) = 0$$

Let $z = \tanh r$, this implies that

$$(1 - z^2)\varphi''(z) - 2z\varphi'(z) + (-\tfrac{1}{4} - \tfrac{(1+4k^2)/4}{1-z^2})\phi(z) = 0$$

This is an associated Legendre ODE with order m and its solution is of the form.

$$\phi(z) = k_1 P^{m}_{-1/2}(z) + k_2 Q^{m}_{-1/2}(z);\ \text{degree} = -1/2,\ m(\text{order}) = \tfrac{1}{2}(1+4k^2)^{1/2}.$$

This implies that

$$u(x,y,t) = e^{(m^2-4)^{-1/2}t/4}\left\{(k_1 P^{m}_{-1/2}(\tanh x) + k_2 Q^{m}_{-1/2}(\tanh x))(\cosh x)^{-1/2}\right\}$$

• *Pseudosphere* or *tractoid* $(f(x) = x)$.

The Eq(6.19) takes a form

$$\phi''(r) + \phi'(r) - k^2\phi(r) = 0$$

This turns out to be linear second ODE whose solution is given by

$$\phi(r) = e^{-r/2}(k_1 e^{rm/2} + k_2 e^{-rm/2}), \quad m = (1+4k^2)^{1/2}$$

This implies that

$$u(x,y,t) = e^{kt-x/2}(k_1 e^{mx/2} + k_2 e^{-mx/2}).$$

• *Torus* $(f(x) = \ln(1+\cos x);\ x \in [0,\pi) \cup (0,2\pi)$

The Eq(6.19) takes the form

$$\phi''(r) - \tan(r/2)\phi'(r) - k^2\phi(r) = 0$$

The substitution of $\phi(r) = \varphi(r)\sec(r/2)$ reduces the above ODE to

$$\varphi''(r) - (k^2 + 1/4)\varphi(r) = 0$$

Solving this ODE and using back substitution gives the solution

$$u(x,y,t) = \begin{cases} e^{kt}\sec(x/2)(k_1\cos mx + k_2\sin mx), & -m^2 = (k^2 - 1/4) \\ e^{kt}\sec(x/2)(k_1 x + k_2), & k^2 = 1/4 \\ e^{kt}\sec(x/2)(k_1 e^{mx} + k_2 e^{-mx}), & m^2 = (k^2 - 1/4) \end{cases}$$

Conclusion

A unified general approach has been adopted to carry out the symmetry analysis of heat and wave equations on a general surface of revolution.

By implementing Lie's classical method and triangulation procedure, a complete classification of the surfaces of revolution is obtained according to the symmetries of the heat and wave equation on these surfaces. In addition many new solutions of heat and wave equations are obtained. Some important results of this work are highlighted below.

- The minimal symmetry algebras for the heat and wave equations on surfaces of revolution are determined. It is found out that in both cases, it is of dimension 3 and similar.

- A complete list of surfaces of revolution admitting larger symmetry algebra is obtained leading to conclusion that for the heat equation, there are surfaces with 4, 5 and 9 dimensional symmetry algebras whereas for the wave equation, there are surfaces with 4, 5 and 11 dimensional symmetry algebras. The corresponding symmetry algebras as well as curvature properties of these surfaces are discussed.

Using carefully chosen subalgebra of the minimal symmetry algebra, we succeed in finding new solutions of heat and wave equations on general surfaces of revolution in terms of the determining function.

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Vitae

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